STUDENTS PROJECT

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M.P.SHAH ARTS AND SCIENCE COLLEGE,

SURENDRANAGAR

T.Y.B.Sc

(2018-19)

A MATHEMATICS PROJECT ON

GRAPH LABELING

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INTRODUCTION OF GRAPH

0.1 HISTORY OF GRAPH THEORY

In 1736,LEONHARD EULER wrote a paper on the seven bridges of Königsberg which is regarded as the first paper history of GRAPH THEORY.A Graph in this context is made up of vertices ,nodes or points which are connected by edges, arcs or lines.

Graph theory is now a major tool in mathematical research, electrical engineering, computer programming and networking, business administrating,sociology,economics, marketing and communications and so on...

There are many research topics in graph theory. Some of major themes in graph theory are Graph coloring , Spanning tree , Planner graphs , Networks , Eulerian tours , Hamiltonian cycle , Matching , Domination theory and " GRAPH LABELING ".

0.2 APPLICATION OF GRAPH

Because of its inherent simplicity, graph theory has a very wide range of application on engineering, in physical, social, and biological sciences, in linguistics, and in numerous other areas. A graph can be used to represent almost any physical situation involving discrete objects and a relationship among them. The following are four examples from hundreds of such applications.

(1) Königsberg bridge problem :-

The Königsberg bridge problem is the best known example in the graph theory.It was solved by LEONHARD EULER in 1736 by mean of graph.

Two islands C and D formed by the PREGEL RIVER in Königsberg in Russia. Where C and D are connected to each other and to the bank A and B with seven bridge as Shown in figure

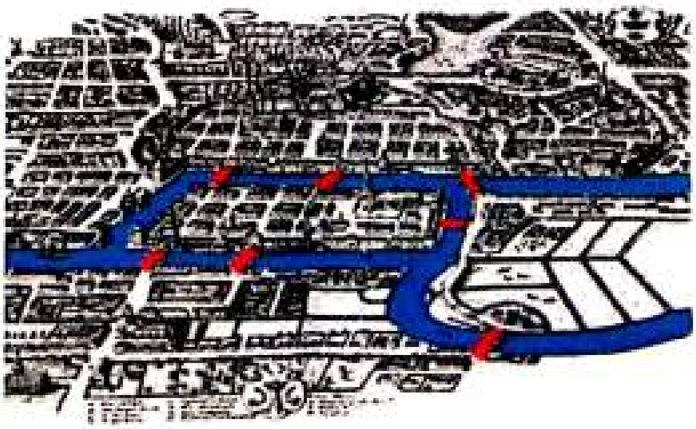

FIG. 0.1. Königsberg bridge problem

The problem was to start at any of the four land area of the city A, B , C or D and walk over each of the seven bridges exactly once and return to starting point.

Euler represent this situation by mean of a graph as shown in figure. The vertices represent the land area and edges represent the land area and edges represent the bridge.

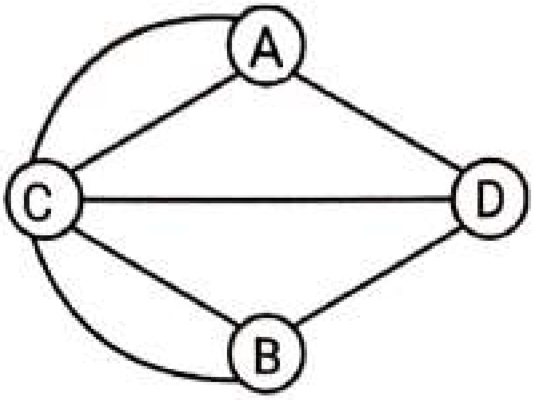

FIG. 0.2.

Now looking at the graph of Königsberg bridge problem,we find that not all its vertices are of even degree.

It is not an Euler graph.

It is not possible to walk over each of 7 bridges exactly once and return to the starting point.

(2) Utilities Problem

(3) Electrical Network Problem

(4) Seating Problem

CHEPTER1

HISTORY OF GRAPH LABELING

1.1 HISTORY

Graph labeling is one of the fascinating areas of graph theory with wide ranging applications. Graph Labeling was first introduced in the 1960's.Most popular graph labeling trace their origin to one introduced by Alex Rosa in 1967. A graph labeling is an assignment of integers to the vertices or the edges, or both, subject to certain conditions. If the domain is the set of vertices we speak about the vertex labeling. If the domain is the set of edges, then the labeling is called the edge labeling. If the labels are assigned to the vertices and also to the edges of a graph, such a labeling is called total.

1.2 DEFINITION AND EXAMPLE

DEFINITION

A "GRAPH LABELING" is an assignment of integers to the vertices or edges or both subject to certain condition.

EXAMPLE

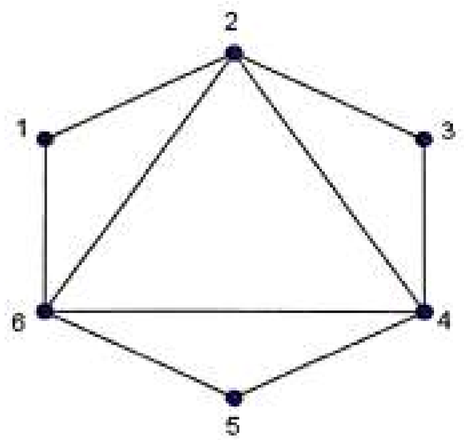

FIG.1.2.1.

1.3 Types of graph labeling :-

There are so many types of graph labeling like...

1) Variation of Cordial labeling :-

- Cordial labeling
- Divisor cordial labeling
- Square divisor cordial labeling
- Product cordial labeling
- Edge product cordial labeling

2) Variation of Prime labeling :-

- Prime labeling
- Vertex prime labeling
- Prime cordial labeling
- Coprime labeling
- Neighborhood prime labeling

3) Variation of Harmonius labeling :-

- Harmonius labeling
- Sequentional harmonius labeling
- Odd harmonius labeling
- Even harmonius labeling
- Strongly c- harmonius labeling

4) Variation of Graceful labeling :-

- Graceful labeling
- α graceful labeling
- γ graceful labeling
- Harmonius graceful labeling
- Odd graceful labeling

5) Variation of Antimagic type labeling :-

- Antimagic labeling
- (a,d) - Antimagic labeling
- (a,d) - Antimagic total labeling
- Face Antimagic labeling
- Product Antimagic labeling
- d- Antimagic labeling of type (1,1,1)

6) Variation of Divisor labeling :-

- Planer zero divisor graph
- Zero divisor graph labeling
- Modular multiplicative divisor labeling

CHAPTER2

CORDIAL GRAPH LALELING

2.1 DEFINITION AND EXAMPLE

The concept of cordial graph labeling was introduced by I.CAHIT in 1996.

DEFINITION

Let mapping f : V(G) → {0,1} is binary vertex labeling of a graph G is called cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is cordial if it admits cordial labeling.

EXAMPLE

A cordial labeling of helm H_3 is shown in below figure -1.

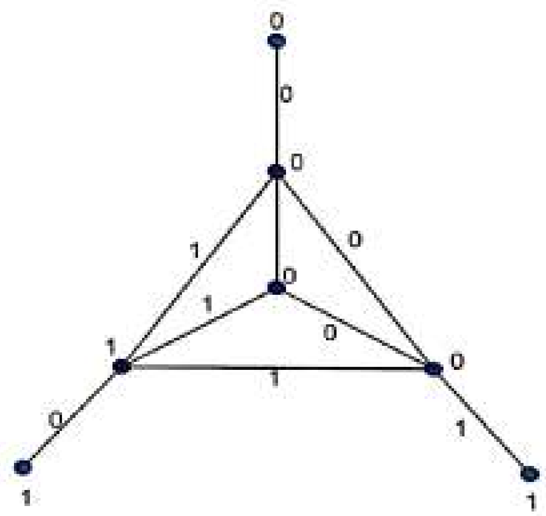

FIG.2.1.1. cordial labeling of helm H_3

The cordial tree , graceful tree and harmonius tree related as follows.

- Any graceful tree is cordial.
- Any harmonius tree is cordial.
- The cycle C_4 is graceful and cordial but not harmonius.
- C_5 is harmonius and cordial but not cordial.
- Complete graph K_5 is nither harmonius ,graceful and cordial.

2.2 KNOWN RESULTS

Cahit has prove that,

1. Every tree is cordial.
2. Complete graph $K_n$is cordial if and only if n≤ 3.
3. Complete bipartite graph $K_{m,n}$are cordial for all m and n.
4. Wheels W_n=C_n+K_1 are cordial if and only if n ≢3(mod4).
5. All fansF_n = P_n+K_1 are cordial.
6. Every path is cordial.
7. Helms, closed helms and generalized helms are cordial.
8. Flower graph (graphs obtained by joining the vertices of degree one of a helm to the central vertex) is cordial.
9. Every broom graph is cordial.
10. cordial labeling for the splitting graph of some standard graphs.

2.3.Cordial labeling for different types of graphs

2.3.1. Cordial labeling of snake related graphs :-

DEFINITION

An *alternate quadrilateral snake A* (QS_n) is obtained from a path$u_1, u_2, \dots, u_n$by joining $u_i, u_{i+1}$to new vertices v_i, w_irespectively and then joining $v_i$andw_i. That is every alternate edge of path is replaced by C_4.

EXAMPLE

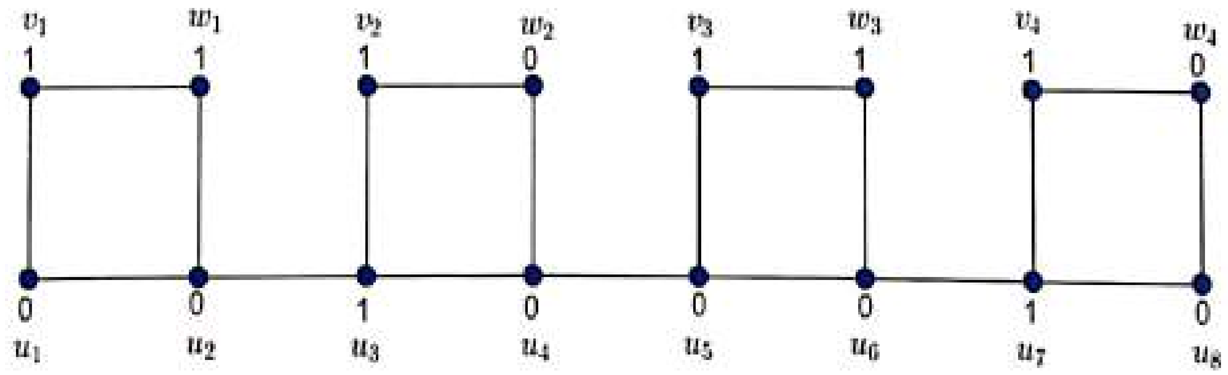

FIG.2.3.1.2. Cordial labeling of *A* (QS_n)

2.3.2.Cordial labeling of splitting graphs :-

DEFINITION

Let G = (V,E) be a graph with $V = S_1 \cup S_2 \cup \ldots \cup S_t \cup T$ where each S_t is the set of vertices having at least two elements and having same degree and

$$T = V \setminus \bigcup_{i=1}^{t} S_i$$

The degree splitting graph of G denoted by DS(G) is obtained from G by adding vertices $W_1, W_2, \ldots, W_t$ and joining W_i to each vertices of $S_i (1 \le i \le t)$.

EXAMPLE

DS(P_n) admits cordial labling . The cordial labeling of DS(P_5) is as shown in below figure.

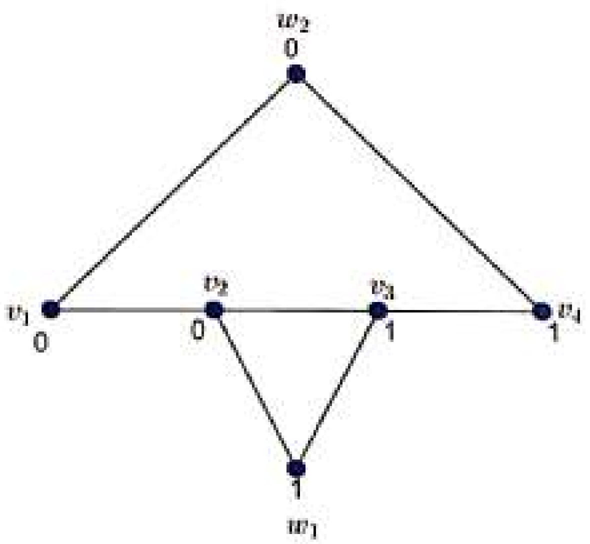

FIG.2.3.2.1. Cordial labeling of DS(P_n)

2.3.3. Cordial graph labeling for Broom graph :-

DEFINITION

Broom graph $B_{n,d}$ is a graph of n vertices which have path P with d vertices and (n-d) pendent vertices all of these being adjacent to either the origin u or the terminus V of the path P.

Now cordial labeling of broom graph.

The following broom graph $B_{10,6}$ has cordial labeling.
Here n=10 and d=6

$$\therefore v_f(1) = {}^{n}/_{2} = 5$$

$$\therefore v_f(0) = {}^{n}/_{2} = 5$$

⇨ $|v_f(1) - v_f(0)| = 0$

And $e_f^*(1) = {}^{n}/_{2} = 5$

$$e_f^*(0) = {}^{n-2}/_{2} = 4$$

⇨ $|e_f^*(1) - e_f^*(0)| = 1$

Hence the graph is cordial.

EXAMPLE

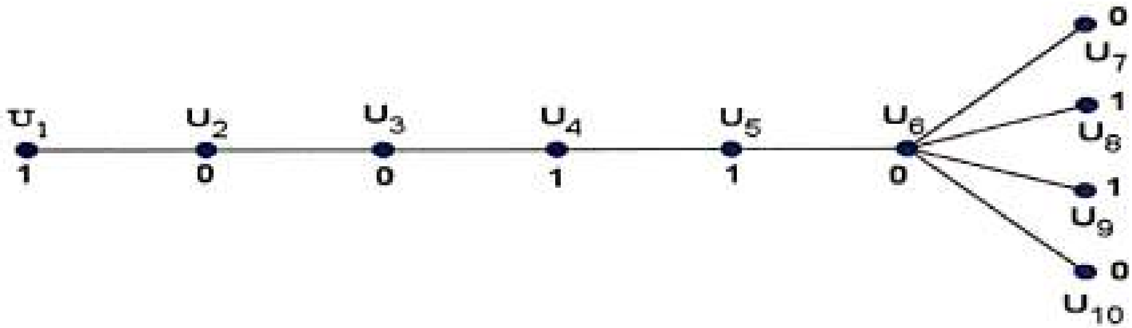

FIG.2.3.3.2. Cordial labeling of $B_{10,6}$

2.3.4. Cordial graph labeling for Flower graph :-

DEFINITION

The Flower graph F_n is the graph obtained from the helm by attaching each pendent edge vertex to the centre vertex of the Wheel(W_n).

Let F_n be the flower graph with n vertices. Let u be the central vertex of F_n. The vertex u is called the hub vertex of the flower graph. Let $u_1, u_2, u_3, \ldots, u_{n-1}, u_n$ be the vertices in the cycle of the flower. Let $v_1, v_2, v_3, \ldots, v_{n-1}, v_n$ be the end vertices of flower.

EXAMPLE

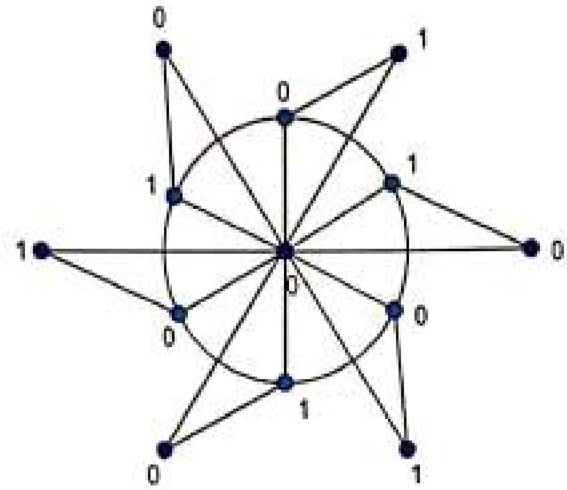

FIG.2.3.4.2. Cordial labeling of F_{13}

2.3.5. Cordial graph labeling for Fan releted graph :-

DEFINITION

The fan $f_n (n \geq 2)$ is obtained by joining all vertices of P_n (Path of n vertices) to a further vertex called the center and contains n+ 1 vertex and 2n-1 edges.

i.e. $f_n = P_n + K_1$.

The graph obtained by joining two copies of fan graph f_n by a part of arbitrary length is admits cordial labeling.

EXAMPLE

The following figure shows the cordial labeling of graph G obtained by joining two copies of fan graph f_5 by a pathP_4.

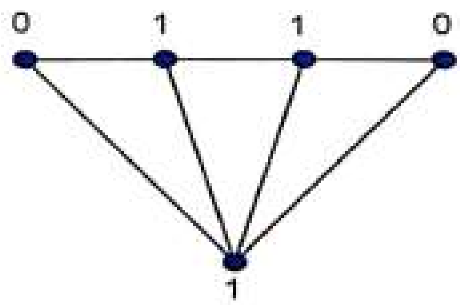

FIG.2.3.5.1. Cordial labeling of f_5

2.3.6. Cordial graph labeling for Ladder graph :-

DEFINITION

The ladder graph L_n is a planar undirected graph with 2n vertices and 3n-2 edges. It is obtained as the cartesian product of two graphs,one of which has only one edge : $L_{n,1} = P_n \times P_1$. where n is the number of rings in the ladder.

EXAMPLE

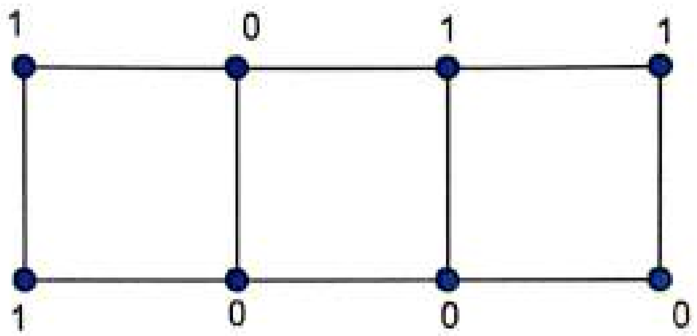

FIG.2.3.6.1. Cordial labeling of L_4

CHAPTER 3

PRIME LABELING

3.1 DEFINITION AND EXAMPLE

The notion of prime labeling originated with Entringer and was considered in a paper by Tout, Dabboucy and Howalla .

DEFINITION

Let G= G(V,E) be a graph. A bijection f: V→{1,2,3,.....,|v|} is called prime labelling if for each e={u,v} belong to E, we have GCD (f(u),f(v))=1. A graph that admits a prime labeling is called a prime graph.

Example

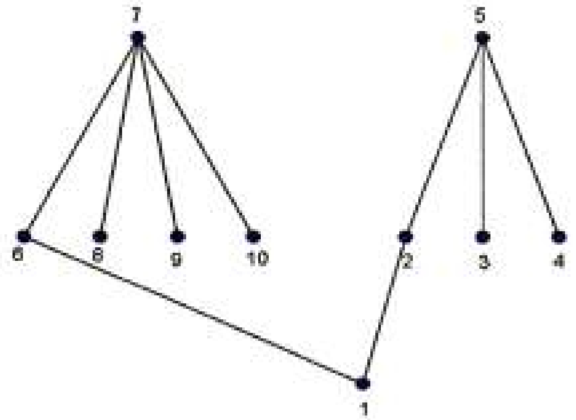

FIG.3.1.1. prime labeling of T_{10}

3.2 KNOWN RESULTS

1. Path P_n on n vertices is prime graph.
2. Cycle C_n on n vertices is prime graph.
3. Wheel W_n is a prime graph if and only if n is even.
4. Complete graph K_n does not have a prime labeling for $n \geq 4$.
5. The graph G obtained by identifying any two vertices of $K_{1,n}$ is a prime graph.
6. Every book graph have prime labeling.
7. The gear graph G_n ,$n \geq 3$ is prime.
8. The ladder graph $L_n = P_2 \times P_n$ has a prime labeling for any integer n ≥ 2.
9. The helm H is prime.
10. Entringer conjectured that every tree is prime for $n \geq 3$.

3.3. Prime labeling for different types of graphs :-

3.3.1. Prime labeling of Book graph:-

DEFINITION

A book graph may be any of servel kinds of graph formed by multiple cycles sharing an edge.

Let $V(B_m) = \{u_0, u_1, u_2, \ldots, u_m, v_0, v_1, v_2, \ldots, v_m\}$.

Define $f: V \to \{1,2, \ldots, |V|\}$ as

$$f(u_0) = 1, \quad f(v_0) = 2,$$

For i= 1, 2,..., m

$f(u_i) = 2(i+1)$And

$$f(v_i) = 2i + 1.$$

Clearly f is a prime labeling for B_m.

Example

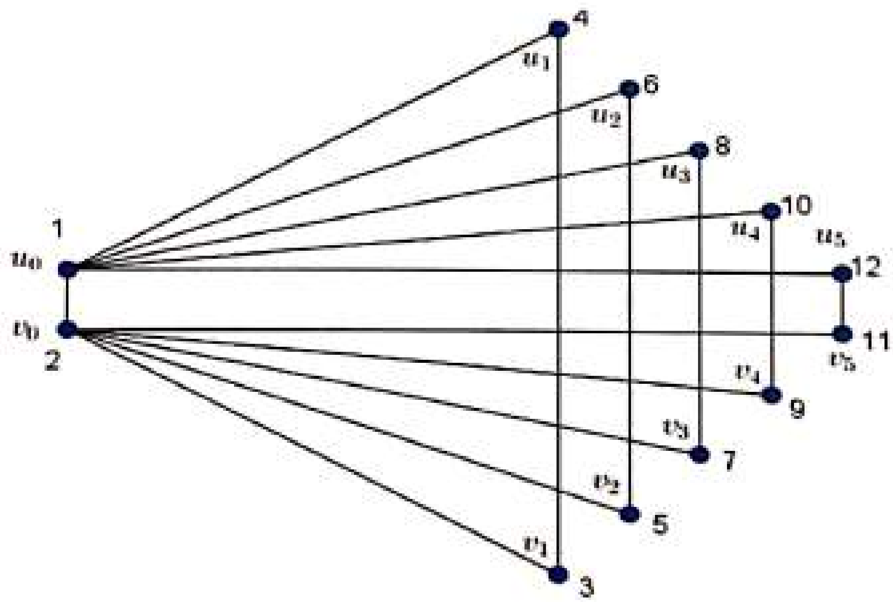

FIG.3.3.1.1. prime labeling ofB_5

3.3.2. Prime labeling of flower graph:-

DEFINITION

The Flower graph Fl_n is the graph obtained from the helm by attaching each pendent edge vertex to the centre vertex of the Wheel(W_n).

Let V be the apex vertex, $v_1, v_2, \dots, v_n$ be the vertices of degree 4 and $u_1, u_2, \dots, u_n$ be the vertices of degree 2 of Fl_n.

Then |V(Fl_n)|= 2n+1 and | E(Fl_n)|= 4n.

We define a prime labelling f: V→{1,2,3,......} given by

f(v) = 1

f(v_i) = 1+ 2i, 1≤i≤n

f(u_i) = 2i, 1≤i≤n.

There exists a bijection f: V→{1,2,3,......,|V|} such that for each e={u,v} belongs to E, we have

GCD (f(u),f(v))=1.

Hence the flower Fl_n admits prime labelling.

Example

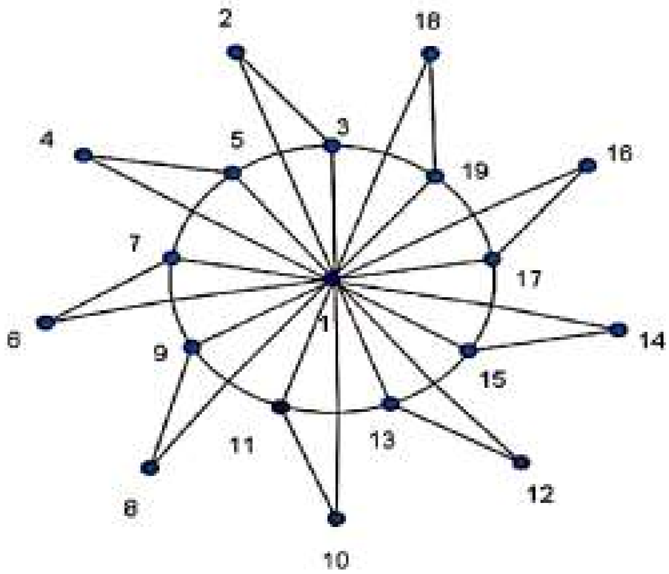

FIG.3.3.2.1. prime labeling of Fl_{19}

3.3.3. Prime labeling of gear graph :-

DEFINITION

The gear graph G_n is a spanning subgraph of W_{2n}obtained by deleting alternate spokes and hence is prime.

The gear graph G_n ,$n \geq 3$ is prime.

Example'

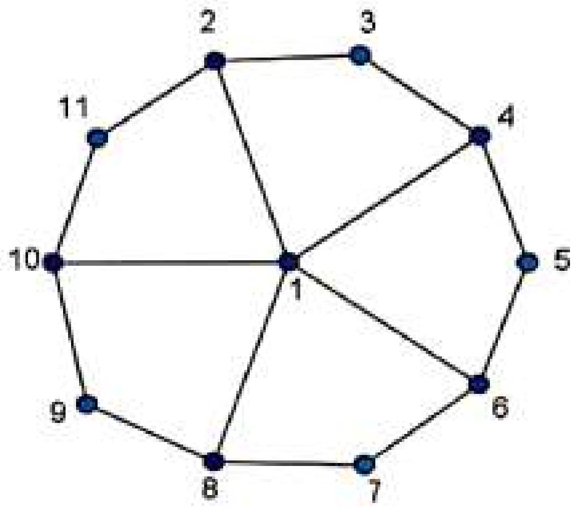

FIG.3.3.3.1. prime labeling of G_{11}

3.3.4. Prime labeling of friendship graph :-

DEFINITION

A friendship graph F_n is a graph which consists of n triangles with a common vertex.

Let $F_n$be the friendship graph with n copies of cycle C_3. Let v' be the apex vertex, $v_1, v_2, \ldots, v_{2n}$ be the other vertices and $e_1, e_2, \ldots, e_{3n}$ be the edges ofF_n.

Define a prime labelling f: V→{1,2,3,......|V|} given by

$f(v') = 1$ $f(v_i) = i+1$ for $1 \leq i \leq n$.

There exists a bijection f: V→{1,2,3,.....,|V|} such that for each e={u,v} belong to E, we have GCD (f(u),f(v))=1.

Hence the friendship graph admits a prime labelling

3.3.3. Prime labeling of gear graph :-

DEFINITION

The gear graph G_n is a spanning subgraph of W_{2n}obtained by deleting alternate spokes and hence is prime.

The gear graph G_n ,$n \geq 3$ is prime.

Example'

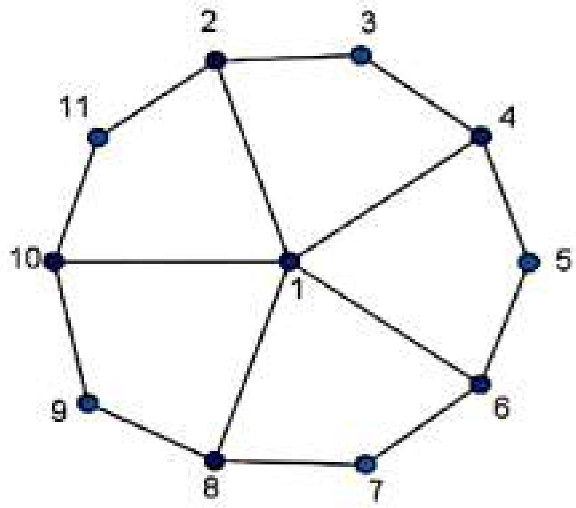

FIG.3.3.3.1. prime labeling of G_{11}

3.3.4. Prime labeling of friendship graph :-

DEFINITION

A friendship graph F_n is a graph which consists of n triangles with a common vertex.

Let $F_n$be the friendship graph with n copies of cycle C_3. Let v' be the apex vertex, $v_1, v_2, \dots, v_{2n}$ be the other vertices and $e_1, e_2, \dots, e_{3n}$ be the edges ofF_n.

Define a prime labelling f: V→{1,2,3,......|V|} given by

f(v') = 1 f(v_i) = i+1 for $1 \leq i \leq n$.

There exists a bijection f: V→{1,2,3,.....,|V|} such that for each e={u,v} belong to E, we have GCD (f(u),f(v))=1.

Hence the friendship graph admits a prime labelling

Example

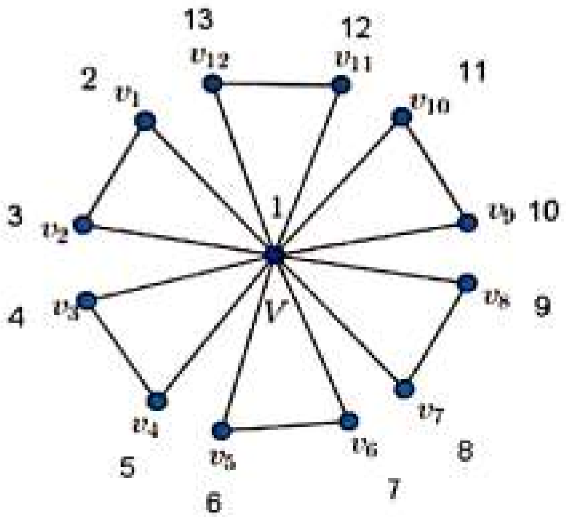

FIG.3.3.4 .1. prime labeling of F_{12}

3.3.5. Prime labeling of helm graph :-

DEFINITION

The helm H_n is a graph obtained from wheel by attaching a pendent vertex at each vertex of the n- cycle .

The helm H_n has 2n+1 vertices and 3n edges. Let w be the central vertex of $H_n$and $u_1, u_2, \ldots, u_n$be the vertices of the wheel's rim. Let V_i be the pendant vertex adjacent to u_i, where u_i is adjacent tou_{i+1}, $1 \leq i \leq n$. That is, $V = \{w, u_i, v_i : i = 1,2, \ldots, n\}$.

Define $f: V \to \{1,,2,3 \ldots, 2n+1\}$ as
f(w) = 1, f(u) = 2, f(v) = 3 and
for i = 1,2,3,...,n,

$$f(u_i) = 2i + 1 , f(v_i) = 2i.$$

Then {f(u) | i = 1,2,3,. . ., n} = {2, 5,7,9,.. . , 2n+1},

$\{f(v_i) | i = 1,2, \ldots, n\}$ = {3, 4, 6, 8, 10 1 . . .,2n} and f is injective.
Now,
GCD $(f(u_i), f(u_{i+1}))$=1 for i = 1, 2, . . . , n-1;
GCD $(f(u_n), f(u_i))$ GCD(2n+1, 2) =1.
Also, GCD(f(w), f(u_i)) = GCD(1, f(u_i)) = 1 and
GCD(f(u_i), f(v_i)) =GCD(2i+1, 2i) = 1 for i = 1, 2, . . . , n.
Hence f is a prime labeling for H_n.

Example

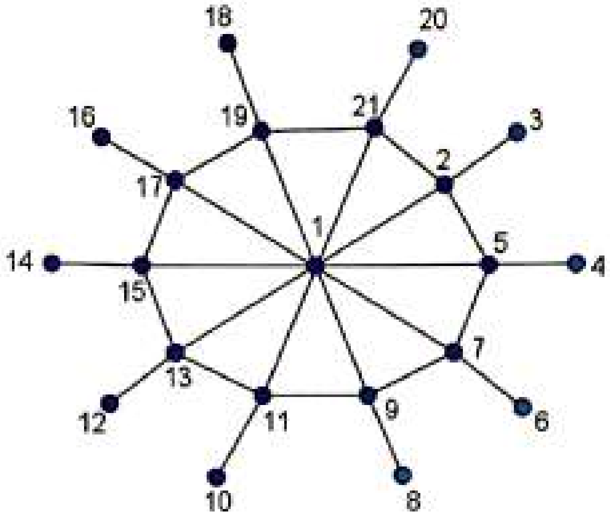

FIG.3.3.5.1. prime labeling of H_{10}

3.3.6. Prime labeling of Ladder graph :-

DEFINITION

The ladder L_n (n ≥ 2) is the product graph $P_2 \times P_n$ which contains 2n vertices and 3n – 2 edges.

The ladder graph $L_n = P_2 \times P_n$ has a prime labeling for any integer n ≥ 2.

$L_n = P_2 \times P_n$ admits a prime labeling if 2n+1 is prime.

Let I = {1, 2, 3, . . ., n}. Let u_i and v_i , i ∈ I, be the vertices of the first row and the second row respectively.

Define f(u_i) = i and f(v_i) = 2n + 1 - i, i∈ I .

Hence f is a prime labeling for L_n.

Example

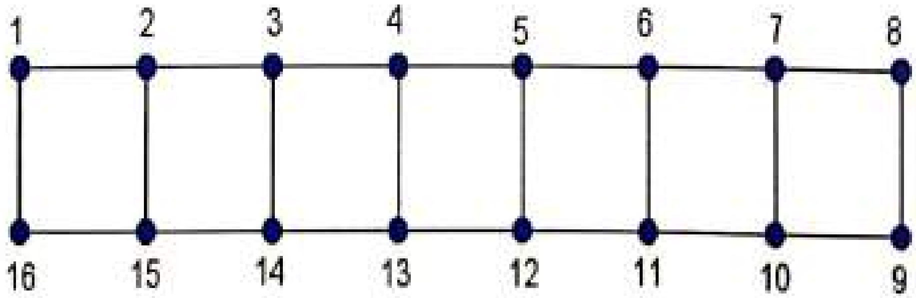

FIG.3.3.6.1. prime labeling of L_8

CHAPTER 4

GRACEFUL LABELING

4.1 DEFINITION AND EXAMPLE

DEFINITION

A function f is called graceful labeling of a graph G with order p and size q if $f: V(G) \to \{1,2,\dots,q\}$is injective and the induced function $f^*: E(G) \to \{1,2,\dots,q\}$defined as $f^*(e = uv) = |f(u) - f(v)|$is bijective. The graph which admits graceful labeling is called a graceful graph.

Example

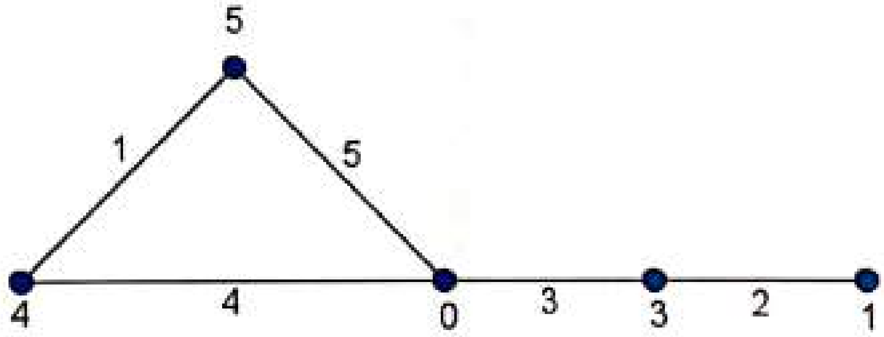

FIG.4.1.1.

The famous Ringel – Kotzig tree conjecture (All trees are graceful) as stated in and many illustrious works on graceful graph brought a tide of different ways of labeling of graph elements such as odd graceful labeling , harmonious labeling etc. Grahan an Sloane introduced harmonious labeling during their study on modular versions of additive bases problems stemming from error correcting codes.

4.2. KNOWN RESULTS

1) The complete graph $K_n$is not graceful for $n \geq 5$.

2) All wheels $W_n$are gracefulfor $n \geq 3$.

3) All trees are graceful.

4) All caterpillars are graceful.

5) Every path is graceful.

6) All fan f_{n-1} are graceful.

7) Gracefulness of union of two path graphs with grid graph and complete bipartite graph.

8) All cycle are graceful except C_5 and C_6.

9) $G = < K_{m_1,n_1} ; ... ; K_{m_t,n_t} >$ the join sum of complete bipartite graphs is graceful , where $m_1 , n_1 , ... , m_t , n_t \in N$.

10) Every Δ_n −snake for $n \equiv 2 or\ 3\ (mod\ 4)$ is almost graceful.

4.3. Graceful labeling for different types of graphs :-

4.3.1.Graceful labeling of cycle graph :-

DEFINITION

For a cycle C_n, each vertices of C_n is replace by connected graph $G_1, G_2, ... G_n$ is known as *cycle of graphs* and it is denoted by $C(G_1, G_2, ... G_n)$. If we replace each vertices by a graph Gi.e. $G_1 = G, G_2 = G, ... G_n = G$, such cycle of graphs is denoted by $C_n(G)$.

Cycle of cycles $C_t(C_n)$, $t \equiv 0$ (mod 2), $n \equiv 0$ (mod 4) is graceful graph.

EXAMPLE:-

C_7 and its graceful labeling shown in following figure.

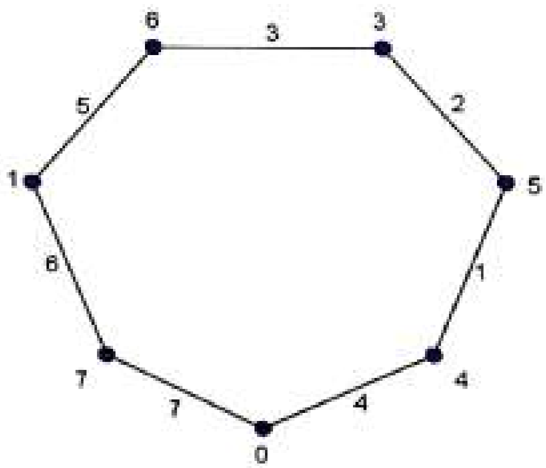

FIG.4.3.1.1.graceful labeling $of\ C_7$

4.3.2. Graceful labeling of snake graph :-

DEFINITION

An alternate triangular snake A(Tn) is obtained from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1}(alternately) to a new vertex v_i. That is every alternate edge of path is replaced by C_3

Example

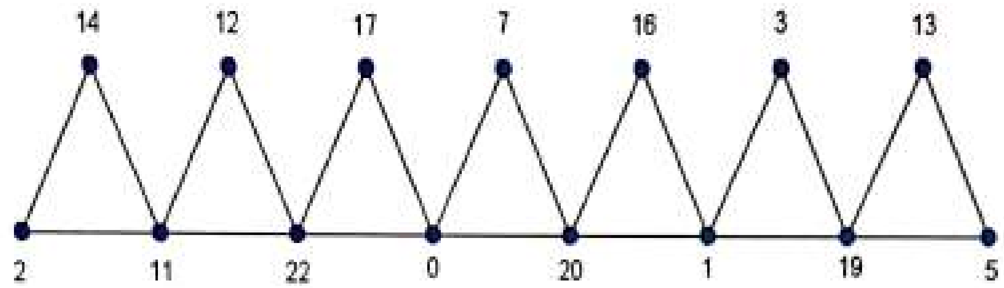

FIG.4.3.2.1.graceful labeling *of* A(Tn)

4.3.3. Graceful labeling of friendship graph :-

DEFINITION

A friendship graph F_n is a graph which consists of n triangles with a common vertex.

A graph Gof size qis odd-graceful, if there is an injection Ø from *V(G)* to {0, 1, 2, ..., 2q-1} such that, when each edge $xy$is assigned the label or weight $|Ø(x) - Ø(y)|$, the resulting edge labels are {1, 3, 5, ..., 2q-1}.

Solairaju and Muruganantham proved that the revised friendship graphs $F(kC_3)$, $F(kC_5)$ and $F(2kC_3)$ are all even vertex graceful, where $k$is any positive integer.

A revised friendship graph $F(kC_n), n \geq 3$ is defined as a connected graph containing $k$copies of $C_n$with a vertex in common.

EXAMPLE

The revised friendship graph $F(kC_n)$is odd graceful, where $k$is any positive integer.

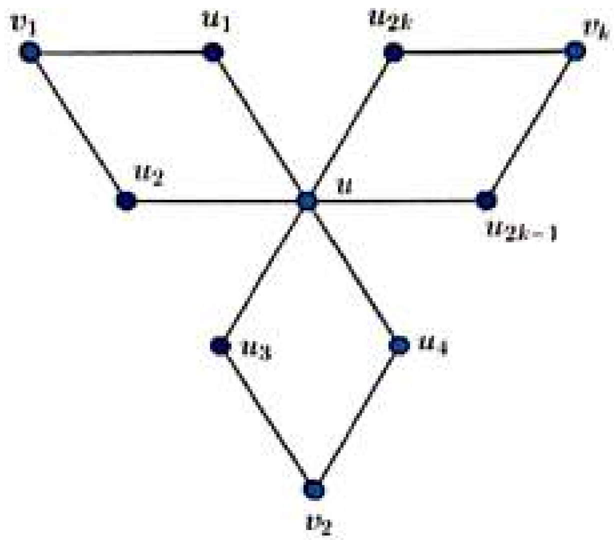

FIG.4.3.3.1.graceful labeling $of\ F(kC_n)$

4.3.4.Graceful labeling of wheel graph :-

DEFINITION

The Wheel graph $W_{1,n}$ $(n \geq 3)$ is a n + 1-vertices graph obtained By connecting all the vertices $\{v_1, v_2, \ldots, v_n\}$ of C_n to the center vertex v, now the vertex set is V $(W_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$. Here v be the center vertex and other vertices $v_1, v_2, \ldots,$ $v_n$be on the rim and the edge set is E$(W_{1,n}) = \{e_1, e_2, e_3, \ldots e_n, e_{n+1}, \ldots, e_{2n}\}$.

Example

$\{v\}$- Center vertex
$\{v_1, v_2, \ldots, v_n\}$- vertices on the rim

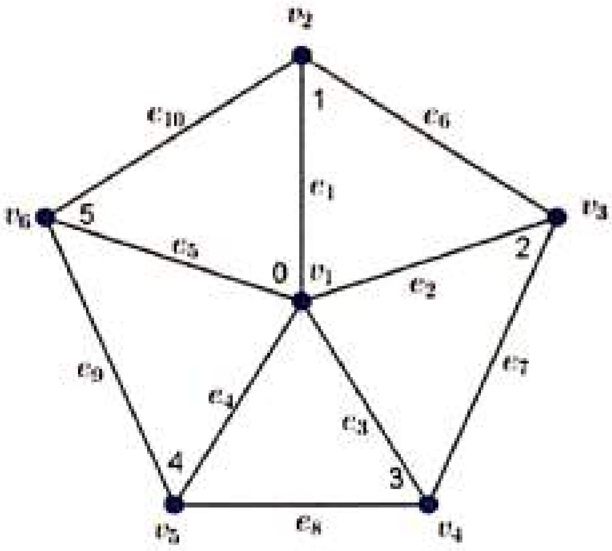

FIG.4.3.4.1.graceful labeling *of* $W_{1,5}$

4.3.5.Graceful labeling of banana tree graph :-

DEFINITION

A banana tree consists of a vertex v joined to one leaf of any number of stars. An example of graceful labeling of banana tree has been illustrated in the figure .

Let $(2K_{1,1}, \dots, 2K_{1,n})$ be the tree obtained by adding a vertex to the union of two copies of each of $K_{1,1}, \dots, K_{1,n}$ and joining it to a leaf of each star. The banana tree obtained in this way is interlaced and therefore graceful. Chen, Lu, and Yeh conjectured in that all banana trees are graceful.
Bhat-Nayak and Deshmukh have constructed three new families of graceful banana

Example

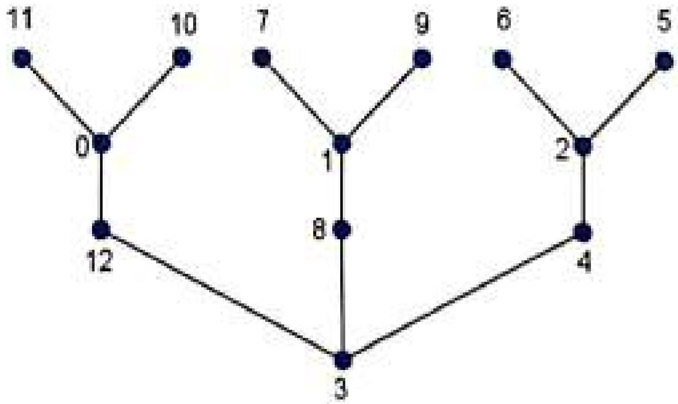

FIG.4.3.5.1.graceful labeling *of* T_{13}

trees using an algorithmic labeling proof. Extending the results of Chen, Lu and Yeh they have shown that the following are graceful.

1. $(K_{1,1}, \ldots, K_{1,t-1} (\alpha+1)K_{1,t}, K_{1,t+1}, \ldots, K_{1,n})$, *where* $0 \leq \alpha < t$;
2. $(2K_{1,1}, \ldots, 2K_{1,t-1} (\alpha+2)K_{1,t}, 2K_{1,t+1}, \ldots, 2K_{1,n})$, *where* $0 \leq \alpha < t$;
3. $(3K_{1,1}, 3K_{1,2}, \ldots, 3K_{1,n})$

Moreover, Murugan and Arumugam showed that any banana tree where all the stars have the same size is graceful by constructing a graceful labeling of these banana trees. Note that a banana tree, in which all the stars have the same size is also a symmetrical tree, so, it is also graceful

4.3.6.Graceful labeling of complete graph :-

DEFINITION

A simple graph in which there exist an edges between every pair of vertices is called complete graph.

A vertex labeling f of a graph G is called graceful if f is an injective mapping from the set of vertices to the set of integers {0,1,...|E(G)|}such that the induced mapping

$$f(xy) = |f(x) - f(y)| \text{, for every } xy \in E(G),$$

assigns different labels to different edges of G. The difference $|f(x) - f(y)|$is called the weight of the edge xy. A graph G is called graceful, if G admits a graceful labeling.

i.e. that the complete graph $K_{2n+1}$is decomposable into 2n+1 sub-graphs that are all isomorphic to a given tree of size n.

Example

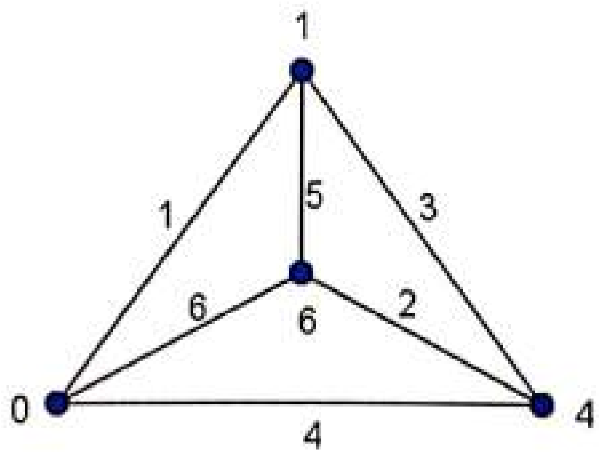

FIG.4.3.6.1.graceful labeling $of\ \ K_4$

CHAPTER5

HARMONIOUS LABELING

5.1 DEFINITION AND EXAMPLE

DEFINITION

A Graph G is said to be *harmonious* if there exist an injection $f: V(G) \to Z_q$ such that the induced function $f^*: E(G) \to Z_q$ defined by $f^*(uv) = (f(u) + f(v))(mod\ q)$ is a bijection and f is said to be harmonious labelling of G.

Example

Harmonious labeling of the graph K_4 is shown in figure.

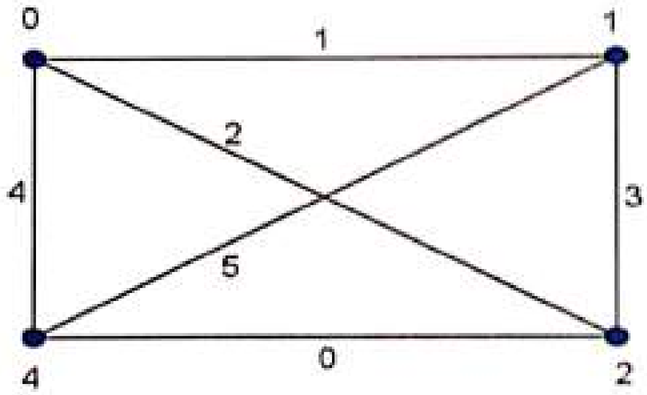

FIG.5.1.1Harmonius labeling $of\ \ K_4$

5.2 KNOWN RESULTS

1) Graham and Sloane conjectured that *Every tree is harmonious.*
   Graham and Sloane also proved that,
2) $K_{m,n}$ is harmonious if and only if $m\ or\ n = 1$.
3) W_n is harmonious $\forall n$.
4) Cycle C_n is harmonious if and only if n is odd.
5) All ladders except L_2 are harmonious.
6) Friendship graph F_n is harmonious except $n \equiv 2(mod\ 4)$.
7) Fan $f_n = P_n + K_1$ is harmonious.
8) For $n \geq 2$ the graph g_n (the graph obtained by joining all the vertices of P_n to two additional vertices) is harmonious.
9) $C_3^{(n)}$ is harmonious if and only if $n \equiv 2(mod\ 4)$.
10) Golomb proved that complete graph is harmonious if and only if $n \leq 4$.

5.3. Harmonious labeling for different types of graphs :-

5.3.1.Harmonious labeling of fan graph

DEFINITION

The fan $f_n(n \geq 2)$ is obtained by joining all vertices of P_n (Path of n vertices) to a further vertex called the center and contains n+ 1 vertex and 2n-1 edges. i.e. $f_n = P_n + K_1$.

Example

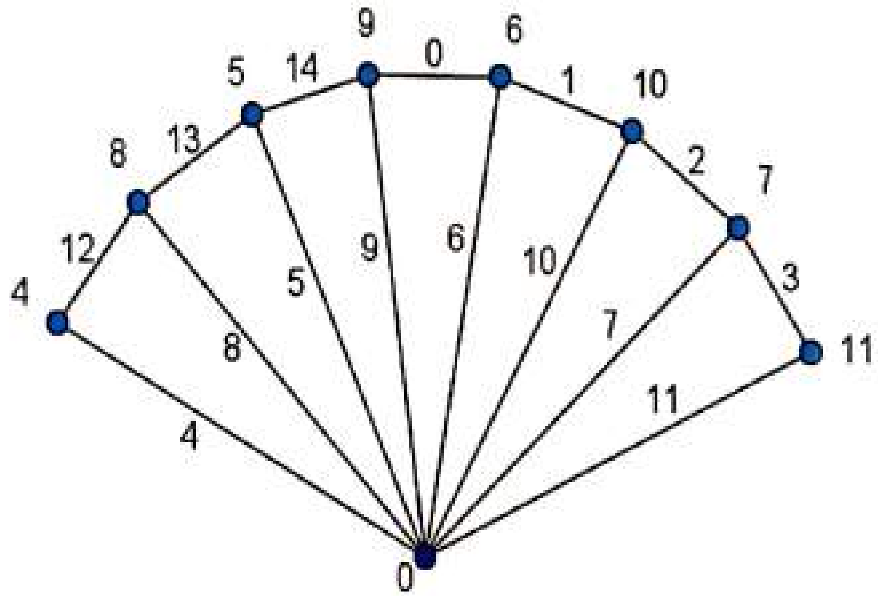

FIG.5.3.1.1Harmonius labeling $of\ \ f_8$

5.3.2.Harmonious labeling of friendship graph :-

DEFINITION

A friendship graph $F_n$is a graph which consists of n triangles with a common vertex.

Example

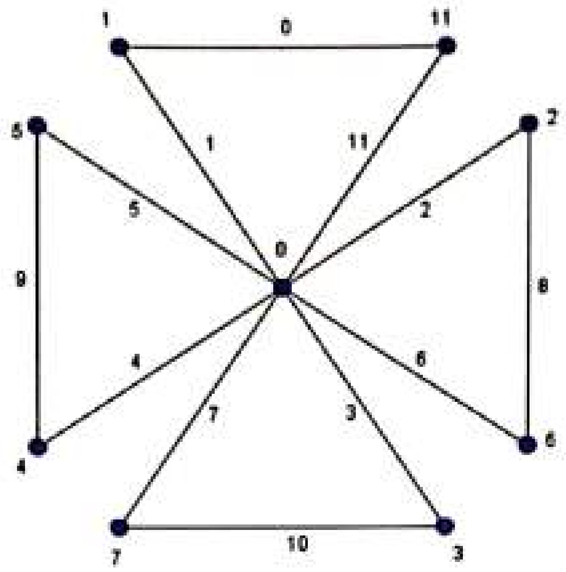

FIG.5.3.2.1Harmonius labeling $of\ F_8$

5.3.3.Harmonious labeling of wheel graph :-

DEFINITION

The wheel graph W_n is defined to be the join of $K_1 + C_n$ i.e. the wheel graph consists of edges which join a vertex of K_1 to every vertex of C_n.

Example

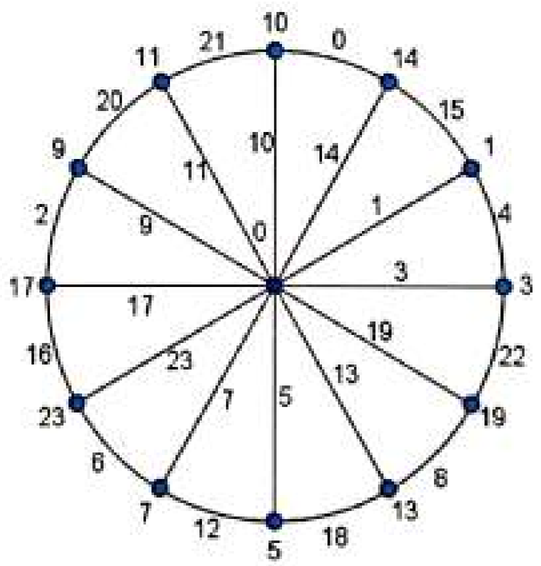

FIG.5.3.3.1Harmonius labeling $of\ W_{12}$

5.3.4.Harmonious labeling of helm graph :-

DEFINITION

The helm H_n is a graph obtained from a wheel by attaching a pendant vertex at each vertex of the n – cycle as shown in figure.

Example

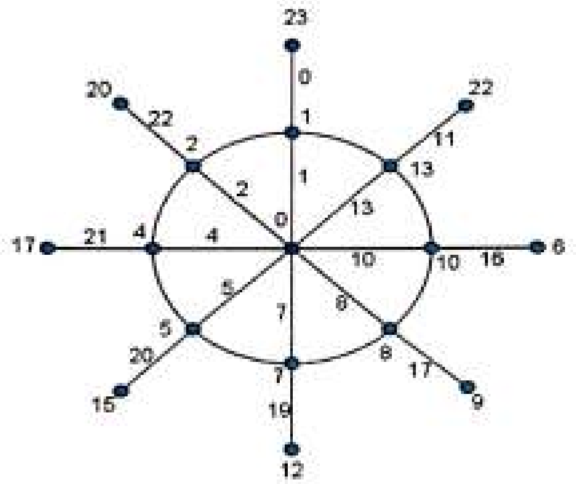

FIG.5.3.4.1 Harmonius labeling *of* H_8

5.3.5.Harmonious labeling of ladders graph :-

DEFINITION

The ladder Ln (n ≥ 2) is the product graph $P_2 \times P_n$ which contains 2n vertices and 3n – 2 edges.

Example

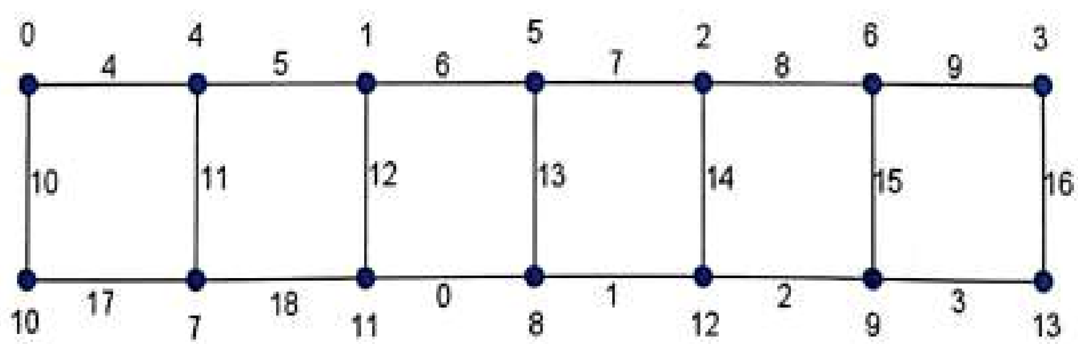

FIG.5.3.5.1 Harmonius labeling *of* L_7

5.3.6.Harmonious labeling of cycle graph :-

DEFINITION

A cycle is a path of edges and vertices where in a vertex is reachable from itself.A closed path is called a cycle.

Example

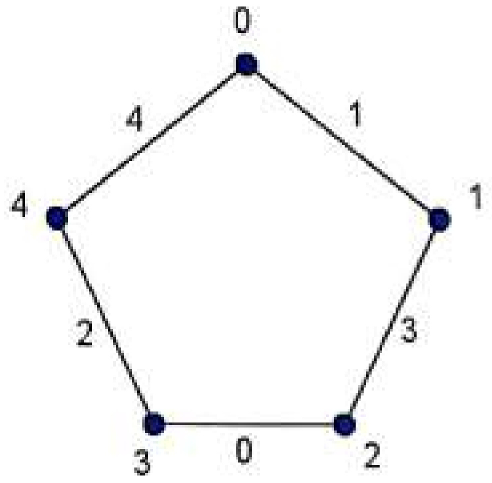

FIG.5.3.6.1Harmonius labeling $of\ C_5$

CHAPTER 6

ANTIMAGIC LABELING

6.1 DEFINITION AND EXAMPLE

DEFINITION

A graph G is called antimagic if the n edges of G can be distinctly labeled 1 through n in such a way that when taking the sum of the edge labels incident to each vertex, the sums will all be different.

Example

For an example of an antimagic labeling for the graph K_4

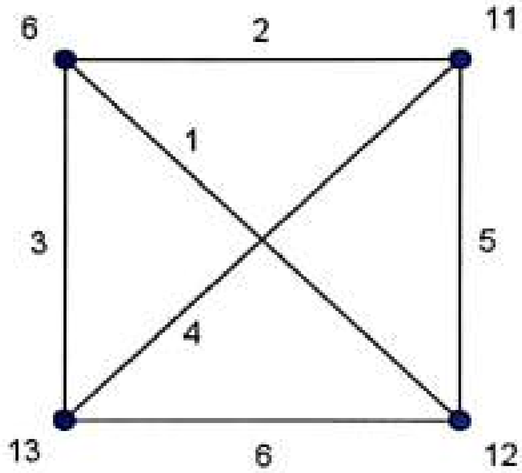

FIG.5.6.1.1Antimagic labeling $of\ K_4$

6.2 KNOWN RESULTS

1)For any integer k $\geq$ 1 , all (2k+ 2) regular graph are antimagic.

2)Every regular bipartite graph with minimum degree 2 is antimagic.

3)$C_n^2$is antimagic and the vertex sums from a set of successive integers when n is odd.

4)Switching graph of a pendent vertex in a path is antimagic.

5)Every even degree regular graph is antimagic.

6)For K $\geq$ 2, every K- regular bipartite graph is antimagic.

7)The path P_{n+1} is antimagic for $n \geq 2$.

8)The cycle C_n is antimagic for $n \geq 3$.

9)Middele graph of path P_n is antimagic.

10)Splitting graph of path P_n is antimagic.

6.3. Antimagic labeling for different types of graphs :-

6.3.1.Antimagic labeling of Caterpillar :-

DEFINITION

A caterpillar is a tree in which all the vertices are within distance 1 of a central path.

Example

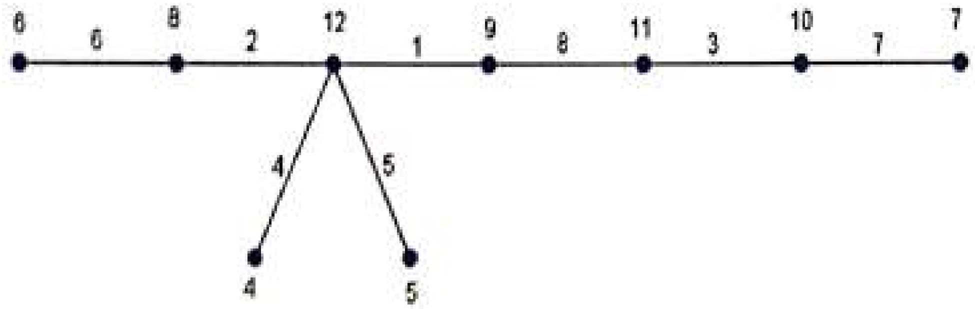

FIG.6.3.1.1Antimagic labeling *of* B_n

6.3.2.Antimagic labeling of Path:-

DEFINITION

A path is walk in which no vertices is repeted .

Example

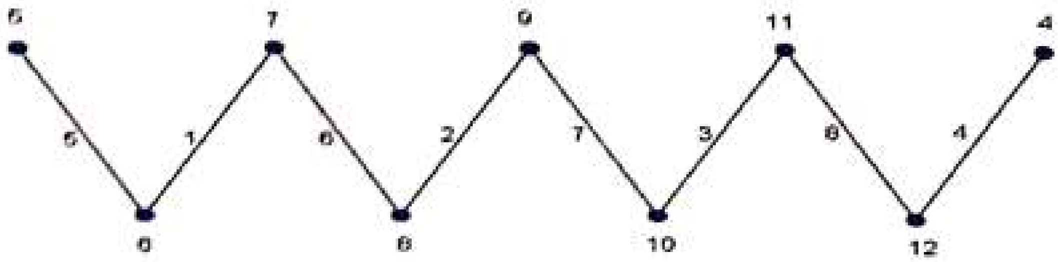

FIG.6.3.2.1Antimagic labeling *of* P_9

6.3.3.Antimagic labeling of Switching graph :-

DEFINITION

A switching graph is an ordinary graph with switches.

Switching of a pendant vertex in a path $P_n$is antimagic.

Let $v_1, v_2, \ldots, v_n$be the vertices of P_n and $G_v$denotes the graph obtained by switching of a pendant vertex $v$of $G = P_n$. Without loss of generality let the switched vertex be v_1. We note that $|V(G_{v_1})| = n$and $|E(G_{v_1})| = 2n$-4. We define $f: E(G_{v_1}) \to \{1,2,\ldots,2n-4\}$as follows:

For $2 \leq i \leq n-1$;

$$f(v_i v_{i+1}) = i-1;$$

For $3 \leq i \leq n$;

$f(v_1 v_i) = n + i - 4$,

Above defined edge labeling function will generate all distinct vertex labels as per the definition of antimagic labeling. Hence the graph obtained by switching of a pendant vertex in a path P_n is antimagic.

Example

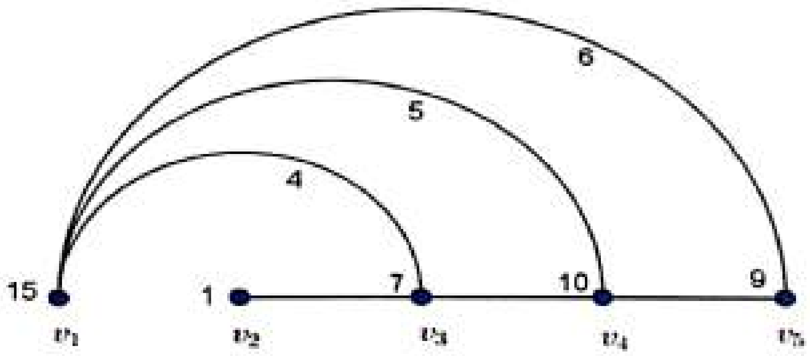

FIG.6.3.3.1Antimagic labeling *of* P_n

6.3.4.Antimagic labeling of Middle graph :-

DEFINITION

The *middle graph M(G)* of a graph *G* is the graph whose vertex set is *V(G)* ∪ *E(G)* and in which two vertices are adjacent if and only if either they are adjacent edges of *G* or one is a vertex of *G* and the other is an edge incident with it.

Example

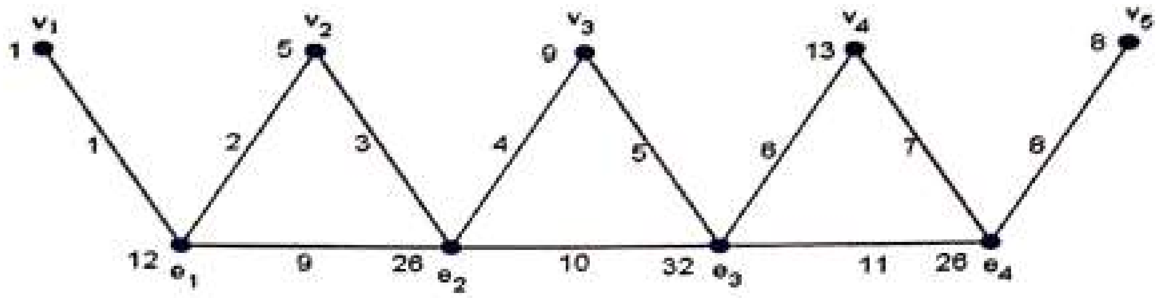

FIG.6.3.2.1Antimagic labeling *of* $M(G)$

6.3.5.Antimagic labeling of cycle graph :-

DEFINITION

A cycle is a path of edges and vertices where in a vertex is reachable from itself.

Example

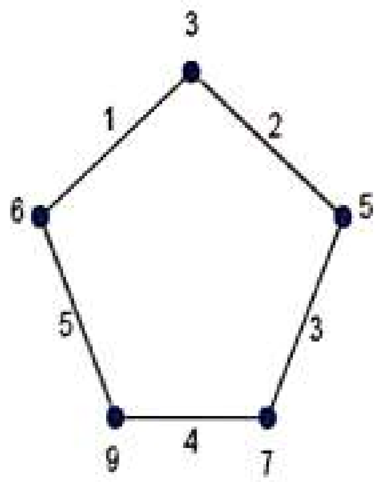

FIG.6.3.5.1Antimagic labeling *of* C_5

6.3.6.Antimagic labeling of Regular graph :-

DEFINITION

A graph in which all vertices are of equal degree is called a regualr graph.

Example

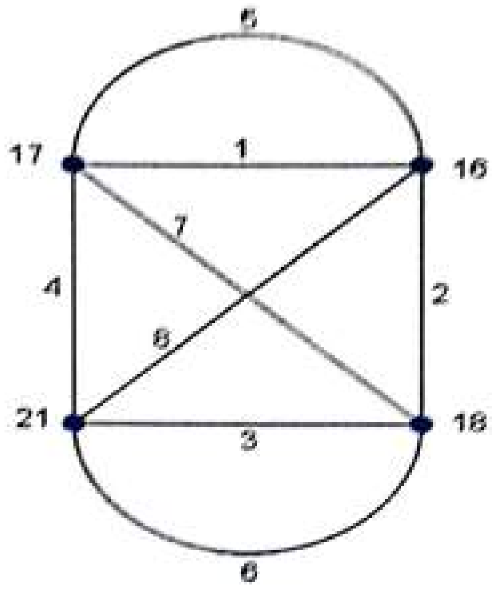

FIG.6.3.6.1Antimagic labeling *of* 4-regular

CHAPTER 7

DIVISOR LABELING

7.1 DEFINITION AND EXAMPLE

DEFINITION :-

In 2000 singh and santhash defined the concept of a divisor graph. They defined a divisor graph G as an ordered pair (V,E) where V⊂ Z and for all u,v ∈ V , u≠v ,uv ∈ E if and only if u|v or v|u.

EXAMPLE :-

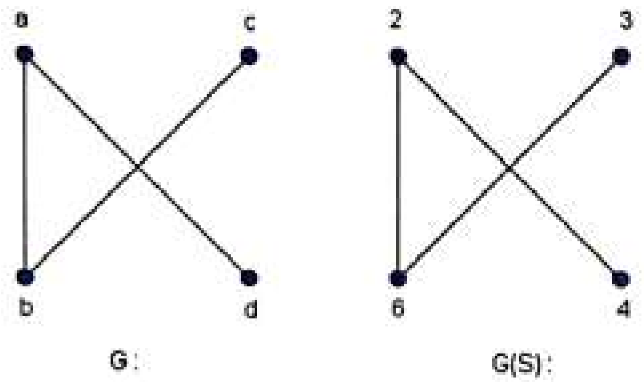

FIG.7.1.1.

7.2 KNOWN RESULTS

1) Every bipartite graph is a divisor graph.
2) All graphs of order less than six are divisor graphs with the exception ofC_5.
3) $P_m \times P_n$ is a divisor graph.
4) The cycles $C_{2n}$are divisor graphs for all $n \geq 2$.
5) Odd cycles C_{2n+1}*for* all $n > 1$ are not divisor graphs.
6) The Petersen graph is not a divisor graph.
7) Every graph is a subgraph of a divisor graph.
8) Every tree is a divisor graph.
9) The ladder graph $L_n$is a divisor graph.
10) Helm graph is divisor graph.

7.3. Divisor labeling for different types of graphs :-

7.3.1.Divisor labeling of ladder graph :-

DEFINITION

The ladder Ln (n ≥ 2) is the product graph $P_2 \times P_n$ which contains 2n vertices and 3n – 2 edges.

Example

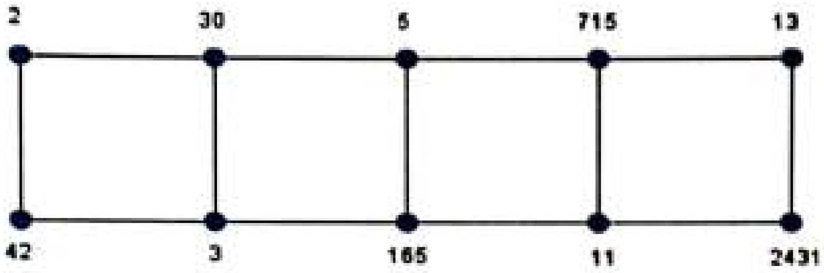

FIG.7.3.1.1Divisor labeling $of\ L_5$

7.3.2.Divisor labeling of wheel graph :-

DEFINITION

The wheelgraph W_n is defined to be the join of $K_1 + C_n$ i.e. the wheel graph consists of edges which join a vertex of K_1 to every vertex of C_n .

Example

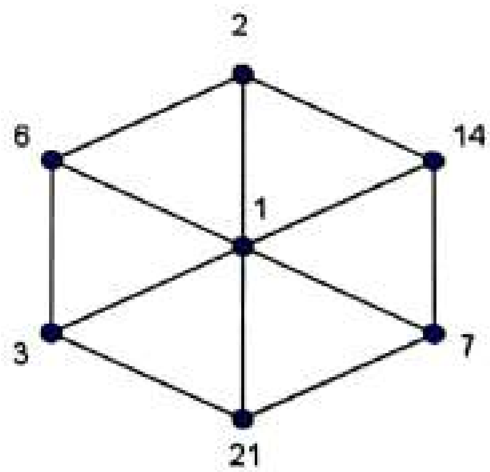

FIG.7.3.2.1Divisor labeling $of\ W_6$

7.3.3.Divisor labeling of helm graph :-

DEFINITION

The helm H_n is a graph obtained from a wheel by attaching a pendant vertex at each vertex of the n – cycle as shown in the Figure.

Example

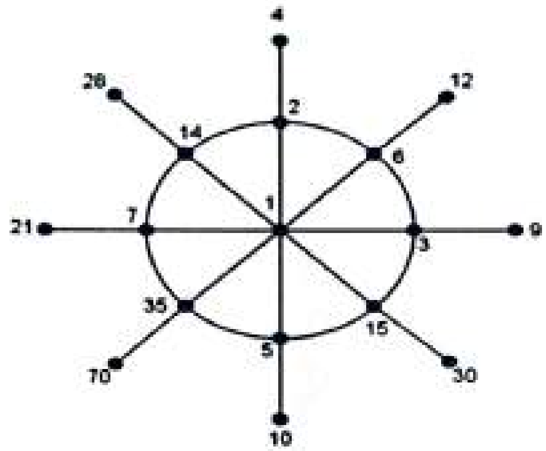

FIG.7.3.3.1 Divisor labeling $of\ H_8$

7.3.4.Divisor labeling of cycle graph :-

DEFINITION

A cycle is a path of edges and vertices where in a vertex is reachable from itself. A closed path is called a cycle.

Example

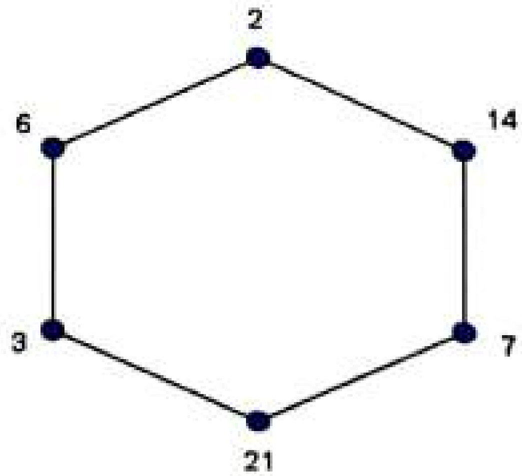

FIG.7.3.4.1 Divisor labeling $of\ C_6$

7.3.5. Divisor labeling of fangraph:-

DEFINITION

A fan is obtained by joining all vertices of P_n to the further vertex called center and contains n + 1 vertex and 2n – 1 edges.

Example

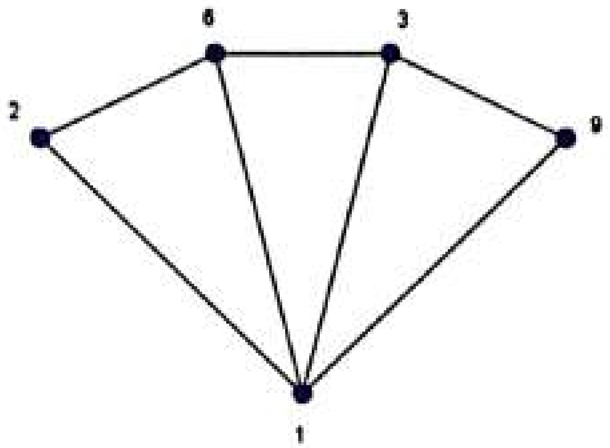

FIG.7.3.5.1Divisor labeling $of\ f_4$

7.3.6. Divisor labeling of friendshipgraph :-

DEFINITION

F_n is a graph which consists of n triangles with a common vertex.

Example

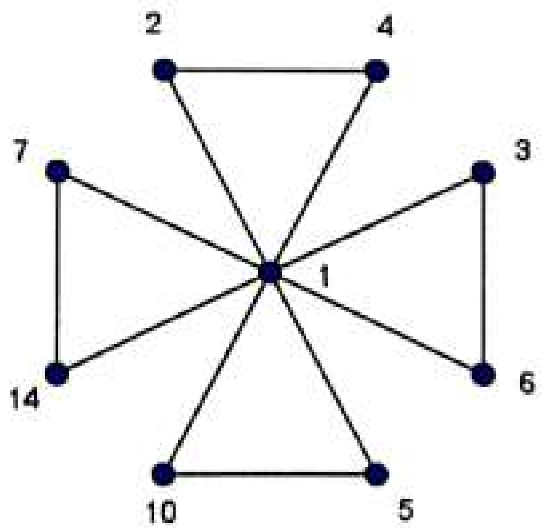

FIG.7.3.6.1Divisor labeling $of\ F_8$

REFERENCE :-

1] Joseph A. Gallian:"A Dynamic Survey Of Graph Labeling"(2017) .

2] I.Cahit, "On cordial and 3-equitable labelings of graph", Utilitas math.

3] T.Nicholas, "Some Labeling Problems in Graph Theory".

4] Sami K.Vaidya, Udayam M.Prajapati, " Some New Results in Prime Graphs", Open Journal of Discrete Mathematics .

5] Michael Jackanich , " Antimagic labeling of Graph" .

6]R.Varatharajan,S.Navaneethakrishnan and K. Nagarajan, "Divisor cordial labeling of graphs", International Journal of Mathematical Combinatorics .

7] R. L. Watson, " A survey on the graceful labeling of graphs".

8] Dushyant Tanna , "Harmonious Labeling of Certain Graphs" .

Shree M P Shah Arts & Science College
Surendranagar

Certificate

This is to Certify that Projec work for the subject of "Golden Ratio" by

Name	Enrollment No.	Roll No.
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2. Makwana Ambaram K.	3101165590	105
3.Jadav Nilesh K.	3101165530	97
4.Gorvadiya Jagdish H.	3101165520	95
5.Danger Raviraj B.	3101165487	91
6.Gohil Jayraj J.	3101165516	93

A Subjected to the project of Mathematics, satisfactorily completed their teamwork in course of B.Sc. (Sem 5^{th} -6^{th}) during the year 2018-19

Date: Project Guide

Examiner's Signature Head of Department

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We thanks all of my friend giving me moral support.

At last we thank all who directly or indirectly helped us in successful completion of this work

Examiner's signature ***Head of department***

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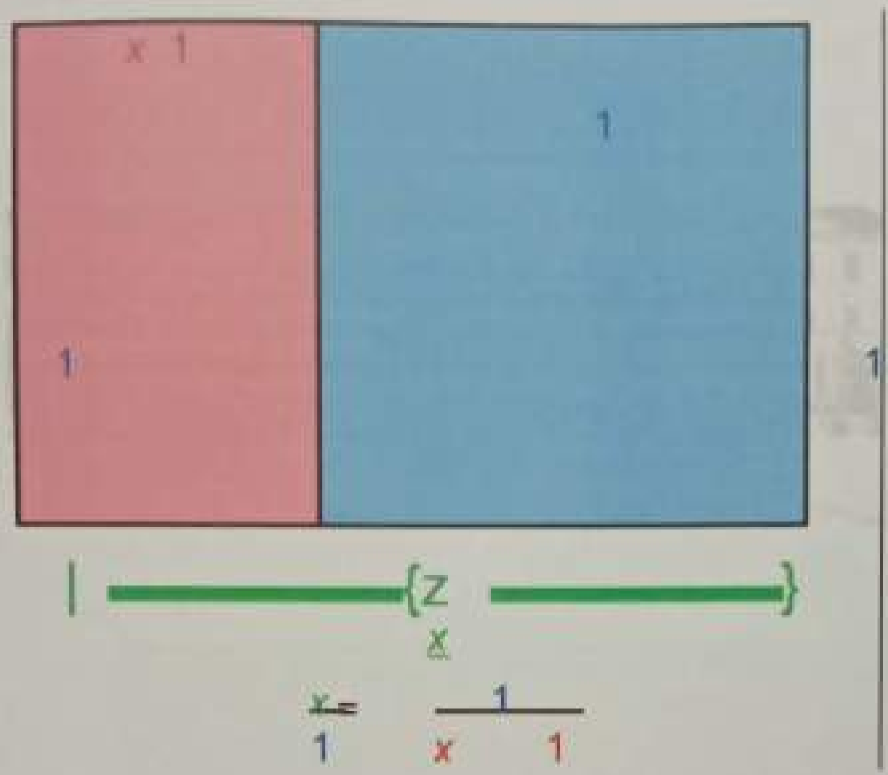
• What is Golden Ratio :-

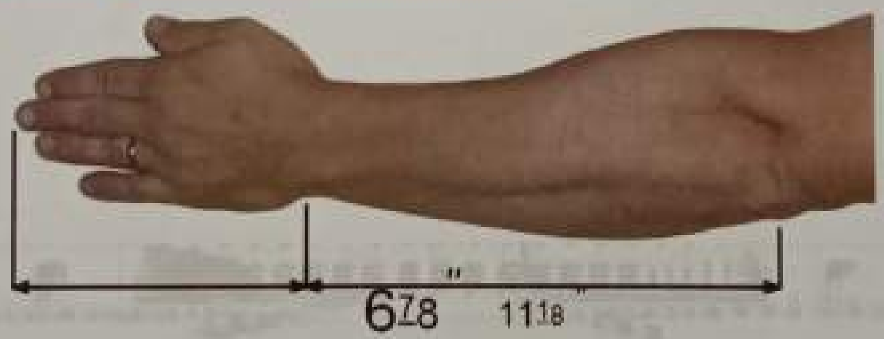
The **Golden ratio** is a special number found by dividing a line into two parts so that the longer part divided by the smaller part is also equal to the whole length divided by the longer part. It is often symbolized using phi, after the 21st letter of the Greek alphabet. Phi **is** usually rounded off to 1.618

EXAMPLE :-

For **example**, the **ratio** of 3 to 5 is 1.666. ... Getting even higher, the **ratio** of 144 to 233 is 1.618. These numbers are all successive numbers in the Fibonacci sequence. These numbers can be applied to the proportions of a **rectangle**, called the **Golden rectangle**.



➢ Human Arm :-



11 1/8 89

6 7/8 = 55

“Make for yourself an ark of gopher wood; you shall make the ark with rooms, and shall cover it inside and out with pitch. This is how you shall make it: the length of the ark three hundred cubits, its breadth **fifty** cubits, and its height **thirty** cubits."

Genesis 6:14-15 (NAS)

$$\frac{50}{30} = \frac{5}{3} \quad 1{:}66$$

African daisy - 21 petals

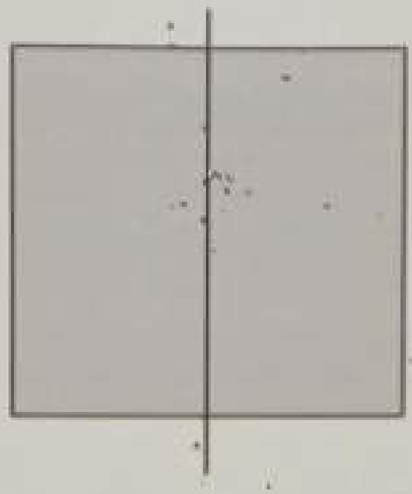
AFRICAN DAISY 21-PETAL

01. Draw a square

This will form the length of the 'short side' of the rectangle.

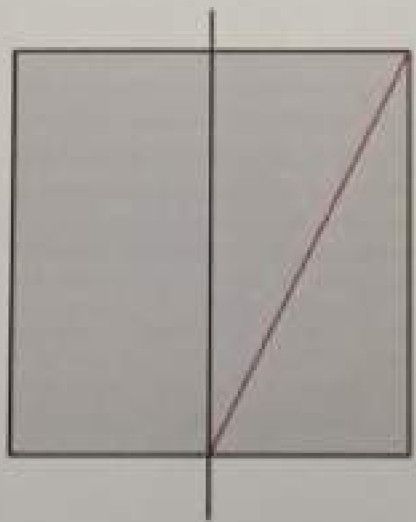
Scanned by CamScanner

02. Divide the square



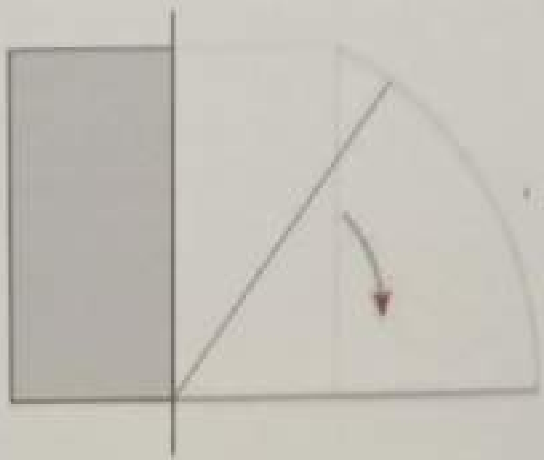
Divide your square in half with a vertical line, leaving you with two rectangles.

03. Draw a diagonal line



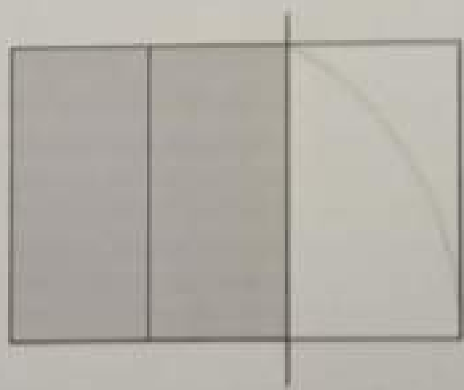
In one rectangle, draw a line from one corner to the opposite corner.

04. Rotate

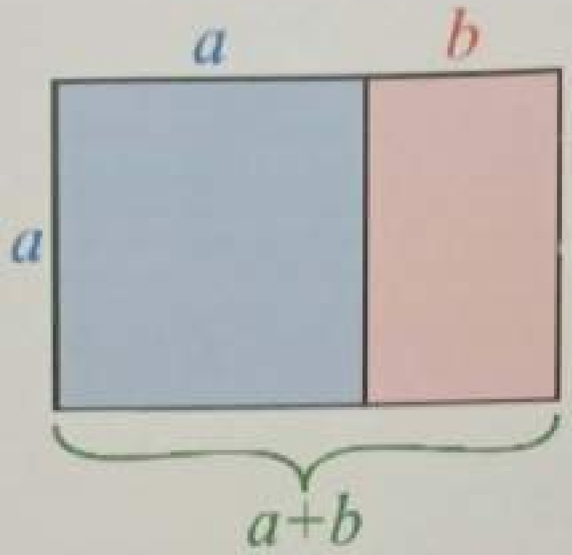


Rotate this line so that it appears horizontally adjacent to the first rectangle.

05. Create a new rectangle



In mathematics, two quantities are in the **golden ratio** if their ratio is the same as the ratio of their sum to the larger of the two quantities. The figure on the right illustrates the geometric relationship. Expressed algebraically, for quantities a and b with $a > b > 0$,



A golden rectangle with longer side ***a*** and shorter side ***b***, when placed adjacent to a square with sides of length ***a***, will produce a similar golden rectangle with longer side ***a*** + ***b*** and shorter side ***a***. This illustrates the relationship .

where the Greek letter phi (φ or ϕ) represents the golden ratio. It is an irrational number with a value of:

The golden ratio is also called the **golden mean** or **golden section** (Latin: *sectio aurea*). Other names include **extreme and mean**

ratio, **medial section**, **divine proportion**, **divine section** (Latin: *sectiondivina*), **golden proportion**, **golden cut**, and **golden number**.

The Actual Value

The Golden Ratio is equal to:

1.61803398874989484820... (etc.)

The digits just keep on going, with no pattern. In fact the Golden Ratio is known to be an Irrational Number, and I will tell you more about it later.

Calculating It

You can calculate it yourself by starting with any number and following these steps:

- A) divide 1 by your number (=1/number)
- B) add 1
- C) that is your new number, start again at A

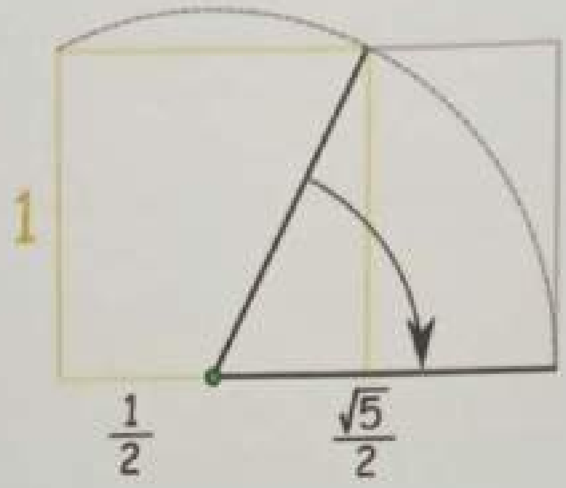
With a calculator, just keep pressing "1/x", "+", "1", "=", around and around. I started with 2 and got this:

Number	1/Number	Add 1
2	1/2=0.5	0.5+1=1.5
1.5	1/1.5 = 0.666...	0.666... + 1 = 1.666...
1.666...	1/1.666... = 0.6	0.6 + 1 = 1.6
1.6	1/1.6 = 0.625	0.625 + 1 = 1.625
1.625	1/1.625 = 0.6154...	0.6154... + 1 = 1.6154...
1.6154...		

It is getting closer and closer!

But it takes a long time to get even close, but there are better ways and it can be calculated to thousands of decimal places quite quickly.

Drawing It



Here is one way to draw a rectangle with the Golden Ratio:

- Draw a square (of size "1")
- Place a dot half way along one side
- Draw a line from that point to an opposite corner (it is $\sqrt{5}/2$ in length)
- Turn that line so that it runs along the square's side
- Then you can extend the square to be a rectangle with the Golden Ratio.

The Formula

That rectangle above shows us a simple formula for the Golden Ratio.

When one side is **1**, the other side is:

$$\varphi = \frac{1}{2} + \frac{\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}$$

The square root of 5 is approximately 2.236068, so the Golden Ratio is approximately (1+2.236068)/2 = 3.236068/2 = 1.618034. This is an easy way to calculate it when you need it.

Interesting fact:the Golden Ratio is equal to 2 × sin(54^0),get your calculator and check.

Fibonacci Sequence

There is a special relationship between the Golden Ratio and the Fibonacci Sequence :

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

(The next number is found by adding up the two numbers before it.)

And here is a surprise: when we take any two successive *(one after the other)* Fibonacci Numbers, **their ratio is very close to the Golden Ratio**.

In fact, the bigger the pair of Fibonacci Numbers, the closer the approximation. Let us try a few:

A	B	B/A
2	3	1.5
3	5	1.666666666...
5	8	1.6
8	13	1.625
...	...	...
144	233	1.618055556...
233	377	1.618025751...
...	...	...

We don't even have to start with **2 and 3**, here I chose **192 and 16** (and got the sequence *192, 16, 208, 224, 432, 656, 1088, 1744, 2832, 4576, 7408, 11984, 19392, 31376, ...*):

A	B	B / A
192	**16**	0.08333333...
16	208	13
208	224	1.07692308...
224	432	1.92857143...

...	...	...
7408	11984	1.61771058...
11984	19392	1.61815754...
...	...	...

The Most Irrational ...

I believe the Golden Ratio is the **most** irrational number. Here is why ...

One of the special properties of the Golden Ratio is that
it can be defined in terms of itself, like this:

$$\varphi = 1 + 1/\varphi$$

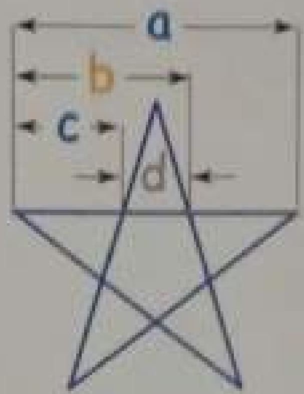
(In numbers: 1.61803... = 1 + 1/1.61803...)

That can be expanded into this fraction that goes on for ever (called a *"continued fraction"*):

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

So, it neatly slips in between simple fractions.

But many other irrational numbers are reasonably close to rational numbers (such as Pi = 3.141592654... is pretty close to 22/7 = 3.1428571...)



Golden ratio conjugate

The conjugate root to the minimal polynomial $x^2 - x - 1$ is

$$-\frac{1}{\varphi} = 1 - \varphi = \frac{1-\sqrt{5}}{2} = -0.61803\,39887\ldots.$$

The absolute value of this quantity (≈ 0.618) corresponds to the length ratio taken in reverse order (shorter segment length over longer segment length, *b/a*), and is sometimes referred to as the *golden ratio conjugate*.[11] It is denoted here by the capital

Phi (Ø):

$$\Phi = \frac{1}{\varphi} = \varphi^{-1} = 0.61803\,39887\ldots.$$

Alternatively, can be expressed as

$$\Phi = \varphi - 1 = 1.61803\,39887\ldots - 1 = 0.61803\,39887\ldots.$$

This illustrates the unique property of the golden ratio among positive numbers, that

$$\frac{1}{\varphi} = \varphi - 1,$$

or its inverse:

$$\frac{1}{\Phi} = \Phi + 1.$$

This means 0.61803...:1 = 1:1.61803....

Alternative forms

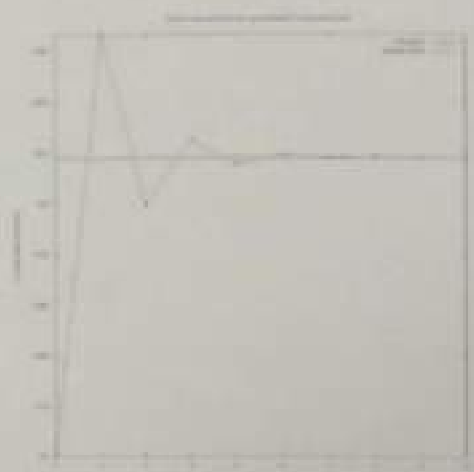
The formula $\varphi = 1 + 1/\varphi$ can be expanded recursively to obtain a continued fraction for the golden ratio.

$$\varphi = [1; 1, 1, 1, \ldots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

and its reciprocal

$$\varphi^{-1} = [0; 1, 1, 1, \ldots] = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

Approximations to the reciprocal golden ratio by finite continued fractions, or ratios of Fibonacci numbers



The formula $\varphi = 1 + 1/\varphi$ can be expanded recursively to obtain a continued fraction for the golden ratio

The convergents of these continued fractions (1/1, 2/1, 3/2, 5/3, 8/5, 13/8, ..., or 1/1, 1/2, 2/3, 3/5, 5/8, 8/13, ...) are ratios of successive Fibonacci numbers.

The equation $\varphi^2 = 1 + \varphi$ likewise produces the continued square root, or infinite surd, form:

$$\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}.$$

An infinite series can be derived to express phi

$$\varphi = \frac{13}{8} + \sum_{n=0}^{\infty} \frac{(-1)^{(n+1)}(2n+1)!}{(n+2)!n!4^{(2n+3)}}.$$

Also:

$$\varphi = 1 + 2\sin(\pi/10) = 1 + 2\sin 18^\circ$$
$$\varphi = \frac{1}{2}\csc(\pi/10) = \frac{1}{2}\csc 18^\circ$$
$$\varphi = 2\cos(\pi/5) = 2\cos 36^\circ$$
$$\varphi = 2\sin(3\pi/10) = 2\sin 54^\circ.$$

These correspond to the fact that the length of the diagonal of a regular pentagon is φ times the length of its side, and similar relations in a pentagram.

Relationship to Fibonacci sequence :-

The mathematics of the golden ratio and of the Fibonacci sequence are intimately interconnected. The Fibonacci sequence is:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987

A closed-form expression for the Fibonacci sequence involves the golden ratio:

$$F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}.$$

The golden ratio is the limit of the ratios of successive terms of the Fibonacci sequence (or any Fibonacci-like sequence), as originally shown by Kepler:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi.$$

In other words, if a Fibonacci number is divided by its immediate predecessor in the sequence, the quotient approximates φ; e.g., $987/610 \approx 1.6180327868852$. These approximations are alternately lower and higher than φ, and converge to φ as the Fibonacci numbers increase, and:

$$\sum_{n=1}^{\infty} |F_n\varphi - F_{n+1}| = \varphi.$$

More generally:

$$\lim_{n\to\infty} \frac{F_{n+a}}{F_n} = \varphi^a,$$

where above, the ratios of consecutive terms of the Fibonacci sequence, is a case when $a = 1$.

Furthermore, the successive powers of φ obey the Fibonacci recurrence:

$$\varphi^{n+1} = \varphi^n + \varphi^{n-1}.$$

This identity allows any polynomial in φ to be reduced to a linear expression. For example:

$$\begin{aligned} 3\varphi^3 - 5\varphi^2 + 4 &= 3(\varphi^2 + \varphi) - 5\varphi^2 + 4 \\ &= 3[(\varphi + 1) + \varphi] - 5(\varphi + 1) + 4 \\ &= \varphi + 2 \approx 3.618. \end{aligned}$$

The reduction to a linear expression can be accomplished in one step by using the relationship

$$\varphi^k = F_k\varphi + F_{k-1},$$

where F_k is the kth Fibonacci number.

However, this is no special property of φ, because polynomials in any solution x to a quadratic equation can be reduced in an analogous manner, by applying:

$$x^2 = ax + b$$

for given coefficients a, b such that x satisfies the equation. Even more generally, any rational function (with rational coefficients) of the root of an irreducible nth-degree polynomial over the rationals can be reduced to a polynomial of degree n □ 1.Phrased in

terms of field theory, if α is a root of an irreducible nth-degree polynomial, then $\mathbb{Q}(\alpha)$ as degree n $\mathbb{Q}$, over , with basis $\{1, \alpha, \ldots, \alpha^{n-1}\}$.

Other Names

The Golden Ratio is also sometimes called the **golden section**, **golden mean**, **golden number**,**divine proportion**, **divine section** and **golden proportion**.

➢ How do you use the golden ratio? :-

One very simple way to apply the **Golden Ratio** is to set your dimensions to 1:1.618. For example, take your typical 960-pixel width layout and divide it by 1.618. You'll get 594, which will be the height of the layout. Now, break that layout into two columns **using** the **Golden Ratio** and voila!

➢ Is Golden Ratio irrational? :-

The **Golden Ratio** is equal to: 1.61803398874989484820... (etc.) The digits just keep on going, with no pattern. In fact the **Golden Ratio** is known to be an **Irrational** Number, and I will tell you more about it later.

➢ What is the exact value of the golden ratio? :-

The Golden Ratio, the perfect number in mathematics, is the squareroot of **5** plus 1, divided **2**. Interestingly, It's the only number that if squared, is equal to itself plus one. In other words, Phi^**2** = Phi+1.

➢ Mona Lisa by Leonardo Da Vinci:-

This picture includes lots of golden rectangles. In below figure, we can draw a rectangle whose bas extends from the woman's right wrist to her left e-below and extend the rectangle vertically until it reaches the very top of her head. Then we will have a golden rectangle. Also, if we draw a squares inside this golden rectangle, we will discover that the edges of these new squares come to all the important focal points of the woman her chin , her eye, her noise and the upturned corner of the mysterious mouth.

It is believed that Leonardo, as a mathematician tried to incorporate of mathematics into art. This painting seems to be made purposefully line up with golden rectangle.

Golden ratio in design ipod:

The ipod was designed by Jonathan Ive and his team of designers. Their goal was to create the perfect product. They achieved this with an extreme amount of attention to detail.One aspect of the design as the basic shape of the device.

The rectangle that is the ipod comes closer than any other MP3 player to the golden ratio 1:1.618 (also sometimes called the golden ratio). This ratio appeals to us at an unconscious level.

It's important because it is found (or appears to be) in so many areas of life, most notably in nature, and most importantly in mathematics. The Fibonacci sequence and the concept of fractals (like the infinity divisible golden rectangle) are great examples of this. Ancient Egyptian and Greek architects built many of their structures with this ratio in mind. Philosophers see this ratio as having an important significance, since it occurs in nature so often.

A lot of people believe that this formula, known as the golden ratio or phi (□) pops up in everyday life.

Golden Ratio in Music :

Music in composed of numeric value and when the Golden Ratio is used to creat a musical piece , it become a living example of Math .The Fibonacci Sequence is also prevent in music.

A few of classical composer used the Golden Ratio ane Fibonacci Sequencing in music piece including Bach, Beethovan , Chopin , and Mozart. Some Modern composers, like Casey ongoven, have explored these age old teuism in the music.

Jame Tennyb redid has piece For Ann. Now it consist of up to twele computer – generated upwardly glissan doing tone having each tone start so it is the golden ratio below the previous tone. So that the combination tone produced by all consecutive tone are a lower or higher pitch already, or soon to be, produced.

Trudi H Garland's points out then on the 5-tone scale (the black note ion the piano) and the 13-notes scale (a complete octave in semitones with the two notes an octave apart include). However, this is bending the truth a litte, since to get both 8 and 13 , we have to count the same note twice (C...C in both cases). It is called an octave. because we usually sing or play the 8th note which complete the cycle by repeating the starting note "an octave higher " and perhaps sounds more pleasing to the ear. But there are really Only 12 different note in our octave, not 13!

Various composers have used the Fibonacci numbers when composing music and some author find the golden section as far back as the Middle Ages(10th century)

GOLDEN RATIO IN VIOLIN CONSTRUCTION

The section on "the violin" in the New Oxford Companion to Music, volume 2, shows how

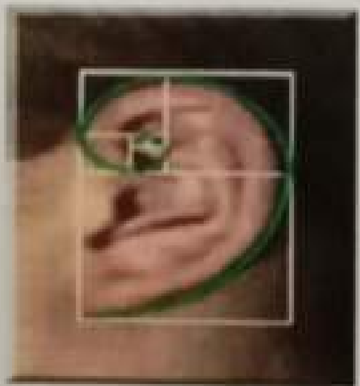
Stradivari was aware of the golden section and used it to place the f-holes in his famous violins.

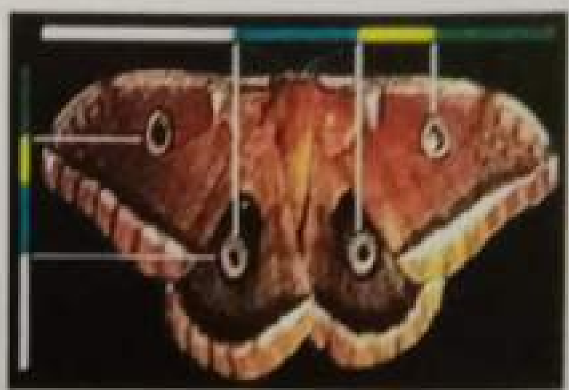
Stradivari used the golden section to place the f-holes in his famous violins.

Baginsky's method of constructing violins is also based on golden sections.

GOLDEN RATIO IN NATURE:

We can found Golden Ratio in our nature at everywhere. For example in humans, animals, flowers, vegetables, fruits and more.





Golden Ratio in Sunflower:

Plants can grow cells in spirals, such as the patterns of seeds in is beautiful sunflower.

The spiral happens naturally because each new cell is formed after a turn.

"new cell, then turn, then another cell, then turn,…"

How far to turn?

In sunflower if any cell don't turn at all then it would have a straight line.

This is because the Golden Ratio (1.61803…0) is the best solution to this problem, and the

sunflower has found this solution in its own natural way.

It should look like this.

➢ Golden Ratio with Spiral Leaf Growth

This interesting behavior is not just found in sunflower seeds. Leaves, branches and petals can growth in spirals, too.

So the new leaves don't block the sun from order leaves, or so that the maximum amount of rain or dew gets directed down to the roots.

➢ Importance of Golden Ratio:-

It's Important because it is found (or appears to be) in so many areas of life, most notably in nature, and most importantly in mathematics. The Fibonacci sequence and the concept of fractals (like the infinitely divisible golden rectangle) are great examples of this. Ancient Egyptian and Greek architects built many of their structures with this ratio in mind. Philosophers see this ratio as having an importance, since it occurs in nature so often.

A lot of people believe that this formula, known as the golden ratio (¢) pops up in everyday life.

➤ Golden Ratio Art Project :-

Using the Golden Ratio or the Fibonacci pattern demonstrate using art, music, nature or architecture, the interesting possibilities of the ratio phi.
Your project will be graded according to these guidelines;

1. Your project will clearly express the Golden Ratio. If it is not visually clear than you will explain your observation of the ratio in written form.

2. Your project will be attractive.

3. Your project will be neat.

4. Your project will have color and/or texture.

5. It will be clear to me that you have given thought and energy to this assignment.

6. No part of your project will be cut and pasted from the internet

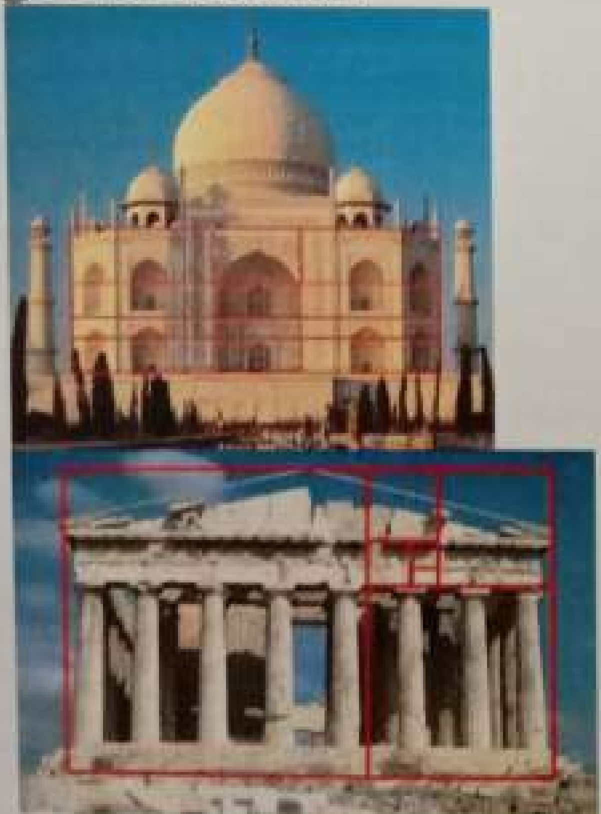
Use Of Golden Ratio :-

- **Use of the golden ratio in other field beside mathematics**

Golden ratio has been used in many other field beside mathematics like ,architecture , art ,painting ,book design ,industrial design.

Architecture :-

The medieval builders of chuches and cathedrals approached the design of their building in much the same way as the Greeks . they tried to connected geometry and art.inside and out , the ir buildings where intricate construction based on the golden section.



Industrial design :-

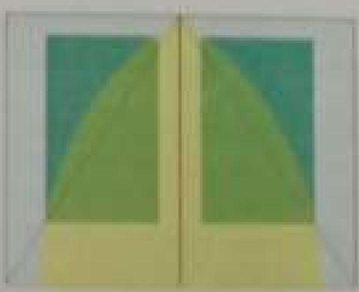
Some sources claim that the golden ratio is commonly used in everyday design ,for example in the shapes of postcards , playing card, posters , wide screen television , photograph and light switch plates

Painting :-

The sixteen century philosopher heinrich Agrippa drew a man over a pentagram inside a circle implying a relationship to the golden ratio.

Book design :-

According to jan tschichold ,there was a time van deviation from the truly beautiful page proportions 2:3, 1:$\sqrt{3}$, and the golden section were rare.many books produced between 1550 and 1770 so this proportions exactly , to within half a millimeter .



USE OF THE GOLDEN RATIO IN EVERYDAY LIFE :-

The golden ratio is very usefull in our life. For example credit cards, logos, design of ipod and more....

Credit cards are in the shape of golden rectangle Standerd sized credit cards are 54mm by 86mm. This creates a ratio of 0.628 which is less than a millimeter off from a perfect golden ratio or golden section of 0.618 , the reciprocal of 1.618

Shree M.P. Shah Arts & Science College, Surendranagar

Affiliated

SAURASHTRA UNIVERSITY

RAJKOT

A

Project Report

on

Z-TRANSFORM

in the subject of

MATHEMATICS

B.Sc. (sem-5 & 6)

Submitted by:

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CERTIFICATE

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under my guidance in fulfilment of the degree of Bachelor of Science in Mathematics (sem-5 & 6) of Saurashtra University, Rajkot during the academic year 2018-2019.

Date : 01/03/2019

Jayshree R. Joshi

Project Guide

Examiner's Signature

Head of the Department

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1.HISTORY

In mathematics and signal processing the z-transform converts a discrete-time signal, which is a sequence of real or complex numbers, into a complex frequency domain representation.

The basic idea known as the z-transform was known to Laplace, and it was re-introduced in 1947 by W.Hurewicz and others as a way to treat sampled-data control systems use with radar. It gives a tractable way to solve linear, constant coefficient difference equations. It was later dubbed "the z-transform" by Ragazzini and Zadeh in the sampled-data control group at Columbia University in 1952.

Z-transform plays the same role in discrete analysis as Laplace transform in continuous systems. As such, Z-transform has many properties similar to those of the Laplace transform. The main difference is that the Z-transform operates not on functions of continuous arguments but on sequence of the discrete integer valued arguments.

The modified or advanced Z-transform was later developed and popularized by E.I.Jury.

2. INTRODUCTION

2.1 Definition

If the function u_n is defined for discrete values (n= 0, 1, 2 ...) and u_n=0 for n<0, then its Z-transform is defined to be

$$Z(u_n)=U(z)= \sum_{n=0}^{\infty} u_n z^{-n} \quad \text{............ (1)}$$

Whenever the infinite series converges

The inverse Z-transform is written as z^{-1}[U (z)] = u_n.

If we insert a particular complex number z into the power series (1) the resulting value of Z (u_n) will be a complex number. Thus the Z-transform $U(z)$ is a complex valued function of a complex variable z.

2.2 Types of Z-transform

- There are two types of z transform:

1. Bilateral Z transform – two sided

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

2. Unilateral z transform – single sided

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

2.3 Formula of Fourier transform and Z-transform

The following eq. (1) and (2) are z-transform and Fourier transform respectively

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad \text{.......... (1)}$$

Replacing z with $e^{j\omega}$ z-transform will become Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad \text{.......... (2)}$$

2.4 Some standard Z-transform

The direct application of the definition gives the following results:

1. $\boldsymbol{Z(a^n)} = \frac{z}{z-a}$

Proof: By definition,

$Z\,(a^n) = \sum_{n=0}^{\infty} a^n z^{-n}$

$= 1 + (a/z) + (a/z)^2 + (a/z)^3 + \ldots\ldots..$

$= \frac{1}{1-(\frac{a}{z})}$

$= \frac{z}{z-a}$

2. $\boldsymbol{Z(n^p) = -z \frac{d}{dz} Z(n^{p-1})}$

Proof:

$Z(n^p) = \sum_{n=0}^{\infty} n^p z^{-n} = z\sum_{n=0}^{\infty} n^{p-1}.n.z^{-(n+1)}$ .... (i)

Changing p to p-1, we get $Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1} z^{-n}$

Differentiating it w.r.t. z

$\frac{d}{dz}[Z(n^{p-1})] = \sum_{n=0}^{\infty} n^{p-1}.(-n).z^{-(n+1)}$ .... (ii)

Substituting (ii) in (i), we obtain $\boldsymbol{Z(n^p) = -z \frac{d}{dz} Z(n^{p-1})}$

Which is the desired recurrence formula.

In particular, we have the following formula:

3. $Z(1) = \frac{z}{z-a}$

4. $Z(n) = \frac{z}{(z-1)^2}$

5. $Z(n^2) = \frac{z^2+z}{(z-1)^3}$

6. $Z(n^3) = \frac{z^3+4z^2+z}{(z-1)^4}$

7. $Z(n^4) = \frac{z^4+11z^3+11z^2+z}{(z-1)^5}$

3. REGION OF CONVERGENCE (ROC)

3.1 Definition

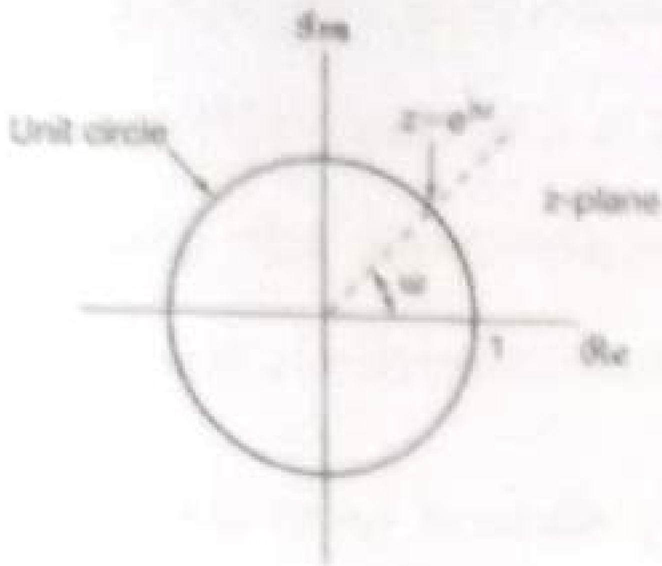
There exists no any point at which the value of function become infinite is known as region of convergence.

As we are aware that the Z- transform of a discrete signal $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

The Z-transform has two parts which are the expression and Region of Convergence respectively.

Whether the Z-transform $X(z)$of a signal $x(n)$ exists or not depends on the complex variable z as well as the signal itself. All complex values of "$z=re^{j\omega}$"for which the summation in the definition converges form a **region of convergence (ROC)** in the z-plane. A circle with r=1 is called unit circle and the complex variable in z-plane is represented as shown below.



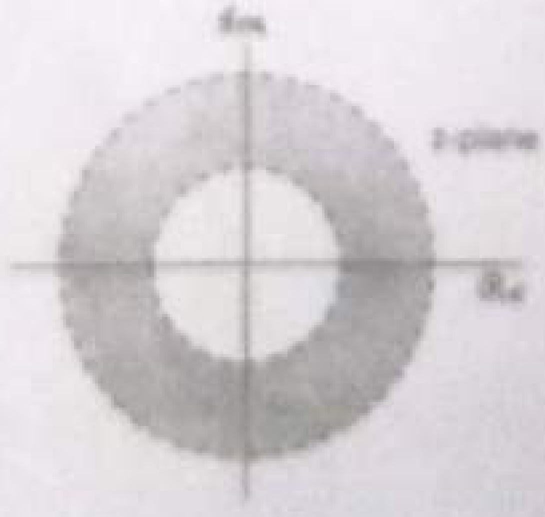
3.2 Properties of ROC

Property 1:

The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.

This property is illustrated in figure below and follows from the fact that the ROC consists of those values of $z = re^{j\omega}$ for which $x(n)r^{-n}$ has a Fourier transform that converges. That is, the ROC of the Z-transform of $x(n)$ consists of the values of z for which $x(n)r^{-n}$ is absolutely summable.

$$\sum_{n=-\infty}^{\infty} |x(n)|r^{-n} < \infty$$



Thus, convergence is dependent only on $r=|z|$ and not on ω. Consequently, if a specific value of z is in the ROC, then all values of z on the same circle (i.e., with the same magnitude) will be in ROC. This by itself guarantees that ROC will consist of concentric rings.

In some cases, the inner boundary of the ROC may extend inward to the origin, and in some cases the outer boundary may extend outward to infinity.

Property 2:

If the Z-transform $X(z)$ of $x(n)$ is rational, then the ROC does not contain any poles but is bounded by poles or extend to infinity.

As with the Laplace transform, this property is simply a consequence of the fact that at a pole $X(z)$ is infinite and therefore does not converge.

Property 3:

If $x(n)$ is of finite duration, then the ROC is the entire z-plane, except possibly $z=0$ and / or $z=\infty$

A finite duration sequence has only a finite number of nonzero values, extending, say, from $n = N$ to $n = M$, where N and M are finite. Thus the

Z-transform is the sum of a finite number of terms; that is

$$X(z) = \sum_{n=N}^{M} x(n)\, z^{-n}$$

For z not equal to zero or infinity, each term in the sum will be finite, and consequently $X(z)$ will converge.

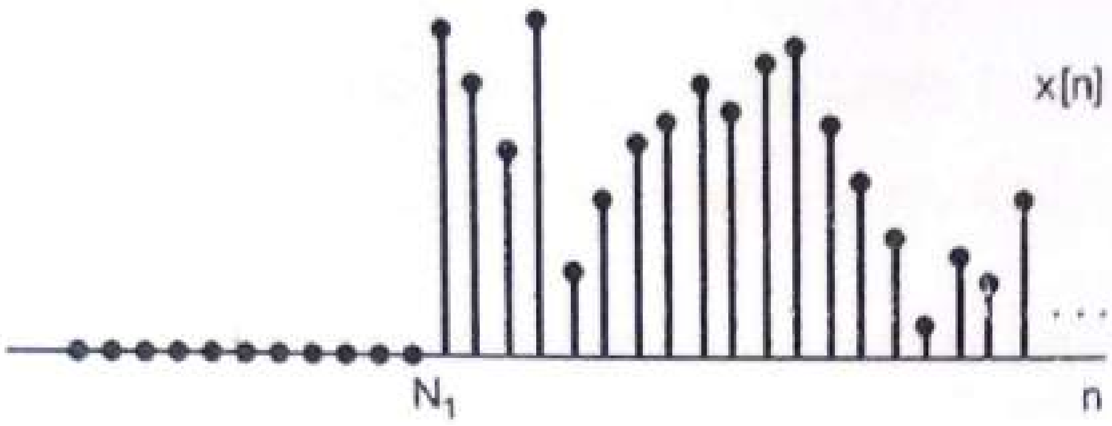
If N is negative and M is positive, so that $x(n)$ has nonzero values both for $n<0$ and $n>0$, then the summation includes terms with both positive and negative powers of z. As $|z| \to 0$, terms involving negative powers of z, become unbounded, and as $|z| \to \infty$, terms involving positive powers of z become unbounded. Consequently, for N negative and M positive, the ROC does not include $z=0$ or $z=\infty$.

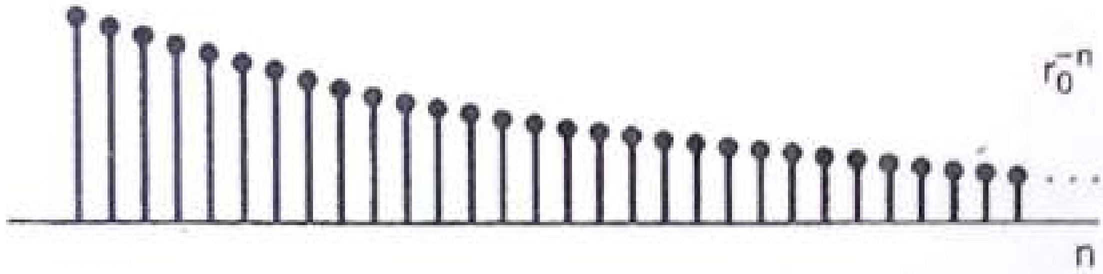
If N is zero or positive, there are only negative powers of z and consequently, the ROC includes $z=\infty$. If M is zero or negative, there are only positive powers of z and consequently, the ROC includes $z=0$.

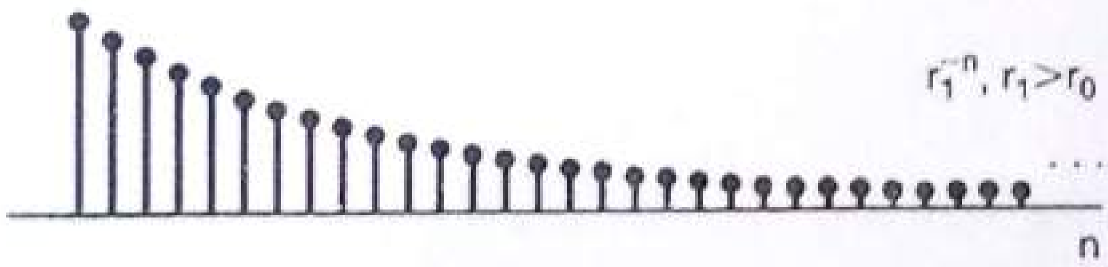
Property 4:

If $x(n)$ is a right sided sequence, and if the circle $|z|=r_0$ is in the ROC, then all finite values of z for which $|z|>r_0$ will also be in the ROC.

The justification for this property follow in a manner identical to that in Laplace transforms. A right sided sequence is zero prior to some value of n, say N_1. If the circle $|z|=r_0$ is in the ROC, then $x(n)r^{-n}$ is absolutely sum able. Now consider $|z|=r_1$ with $r_1>r_0$, so that r_1^{-n} decays quickly than r_0^{-n} for increasing n as illustrated in the figure below.







Consequently, $x(n)\, r_1^{-n}$ is absolutely summable.

For right sided sequences in general $X(z) = \sum_{n=N_1}^{\infty} x(n)z^{-n}$, where N_1 is finite and may be positive or negative.

If N_1 is negative, then the summation above includes terms with positive powers of z, which become unbounded as $|z| \rightarrow \infty$. Consequently, for right sided sequences in general, ROC will not include infinity.

However, for causal sequences, i.e., sequences that are zero for n<0, N_1 will be non-negative, and consequently, the ROC will include $z=\infty$.

Property 5:

If $x(n)$is a left sided sequence, and if the circle $|z|= r_0$ is in the ROC, then all values of z for which $0 <|z|< r_0$ will also be in the ROC.

For left sided sequences, the summation for the Z-transform will be of the form

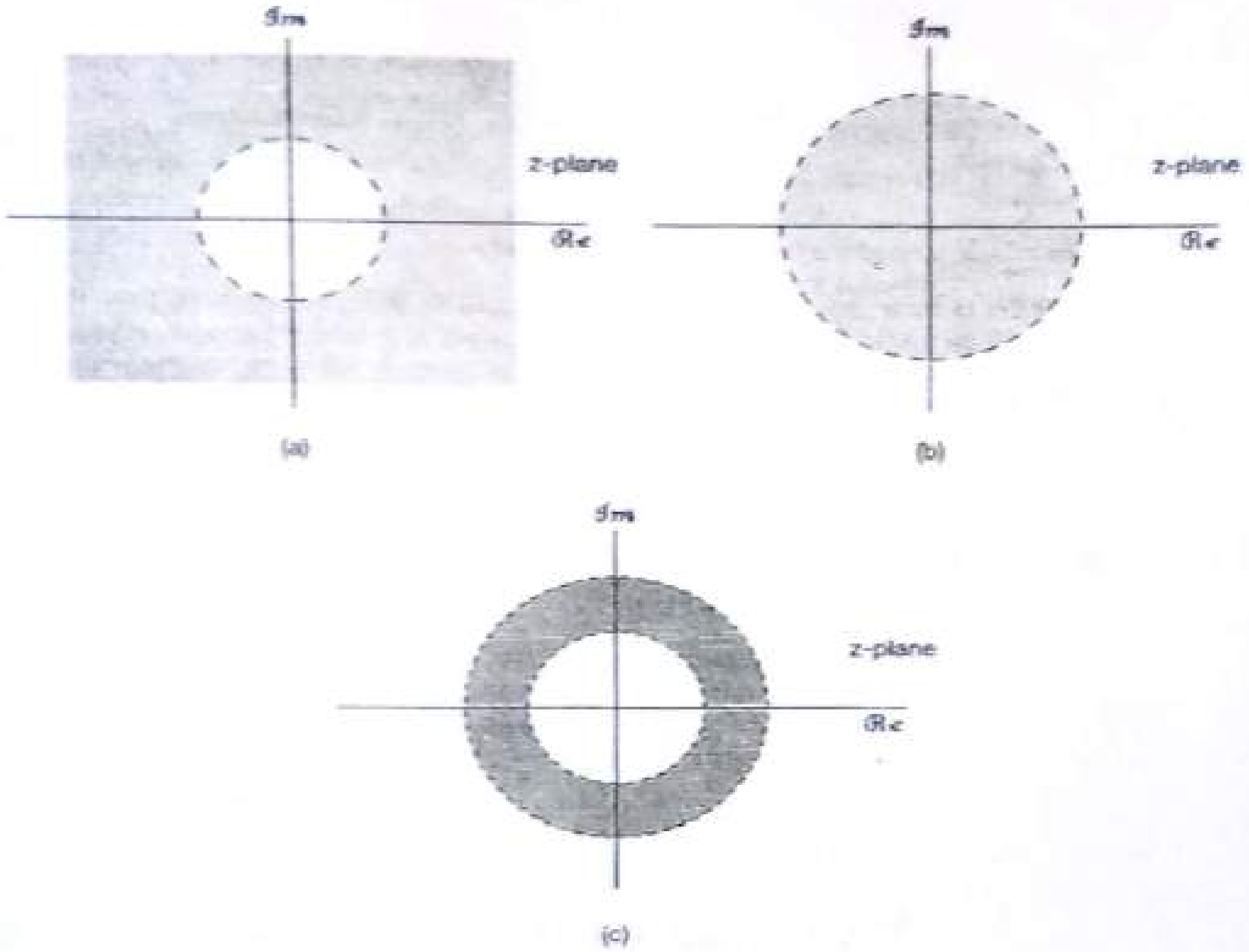
$$x(z) = \sum_{n=-\infty}^{M} x(n)\, z^{-n}$$

Where M may be positive or negative. If M is positive, then the transform includes negative powers of z, which become unbounded as $|z| \rightarrow 0$. Consequently, for left-sided sequences, the ROC will not include $|z|=0$. However, if $M \leq 0$ (so that $x(n)=0$ for all $n>0$), the ROC will include $z=0$.

Property 6:

If $x(n)$is two sided, and if the circle $|z|=r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z|=r_0$.

Like corresponding property in Laplace transforms, the ROC of a two-sided signal can be examined by expressing $x(n)$as the sum of a right-sided and a left-sided signal. The ROC for the right-sided component is a region bounded on the inside by a circle and extending outward to (and possibly including) infinity as in figure (a). The ROC for the left sided component is a region bounded on the outside by a circle and extending inward to, and possibly including, the origin as in figure (b). The ROC for the composite signal includes the intersection of the ROCs of the components as in figure (c).



Property 7:

If the Z-transform $X(z)$ of $x(n)$ is rational, and if $x(n)$ is right sided, then the ROC is the region in the Z-plane outside the outermost pole i.e., outside the circle of radius equal to the largest magnitude of the poles of $X(z)$.

Property 8:

If the Z-transform $X(z)$ of $x(n)$ is rational, and if $x(n)$ is left sided, then the ROC is the region in the Z-plane inside the innermost pole i.e., inside the circle of radius equal to the smallest magnitude of the poles of $X(z)$ other than any at z=0 and extending inward to and possibly including z=0.

4. PROPERTIES OF Z-TRANSFORM

4.1 Linearity Property:

Theorem:

If $x_1(n) \overset{z}{\leftrightarrow} X_1(z)$ with ROC = R_1 and $x_2(n) \overset{z}{\leftrightarrow} X_2(z)$ with ROC = R_2

Then, a $x_1(n)$ +b$x_2(n) \overset{z}{\leftrightarrow} X_1(z)+bX_2(z)$, with ROC containing $R_1 \cap R_2$.

Proof:

Taking the z-transform

$$Z\{ ax_1(n)+bx_2(n)\} = \sum_{n=-\infty}^{\infty}\{ax_1(n) + bx_2(n)\}z^{-n}$$

$$= a\sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + b\sum_{n=-\infty}^{\infty} x_2(n)z^{-n}$$

$$= aX_1(z) + bX_2(z)$$

The ROC of the Linear combination is at least the intersection of R_1 and R_2. For sequences with rational z-transforms, if the poles of $aX_1(z) + bX_2(z)$consist of all the poles of $X_1(z)$ and $X_2(z)$, indicating no pole-zero cancellation, then the ROC will be exactly equal to the overlap of the individual regions of convergence.

If the Linear combination is such that some zeroes are introduced that cancel poles, then the ROC may be larger.

4.2 Time shifting property:

Theorem:

If $x(n) \overset{z}{\leftrightarrow} X(z)$ with ROC= R

Then, $x(n-m) \overset{z}{\leftrightarrow} z^{-m}X(z)$ with ROC= R, except for the possible addition or deletion of the origin or infinity.

Proof:

$Z\{x(n-m)\} \sum_{n=-\infty}^{\infty} x(n-m)\, z^{-n}$

Let n-m=p

$= \sum_{p=-\infty}^{\infty} x(p)\, z^{-(p+m)}$

$= z^{-m} \sum_{p=-\infty}^{\infty} x(p)\, z^{-p}$

$= z^{-m}X(z)$

4.3 Scaling in the Z -Domain:

Theorem:

If $x(n) \overset{z}{\leftrightarrow} X(z)$ with ROC= R.

Then, $z_0^n x(n) \overset{z}{\leftrightarrow} X(\frac{z}{z_0})$ with ROC= $|z_0|$R where, $|z_0|$R is the scaled version of R.

Proof:

$Z\,\{z_0^n x(n)\} = \sum_{n=-\infty}^{\infty} z_0^n x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n) (\frac{z}{z_0})^{-n} = X(\frac{z}{z_0})$

4.4 Time reversal property:

Theorem:

If $x(n) \overset{z}{\leftrightarrow} X(z)$ with ROC= R then $x(-n) \overset{z}{\leftrightarrow} X(\frac{1}{z})$ with ROC=$\frac{1}{R}$.

Proof:

$Z\{x(-n)\} \sum_{n=-\infty}^{\infty} x(-n)\, z^{-n}$

Let -n=p

$$= \sum_{p=-\infty}^{\infty} x(p)\,(z)^{p} = \sum_{p=-\infty}^{\infty} x(p)\,(z^{-1})^{-p} = X(\frac{1}{z})$$

4.5 Convolution property:

Theorem:

If $x_1(n) \overset{z}{\leftrightarrow} X_1(z)$with ROC = R_1& $x_2(n) \overset{z}{\leftrightarrow} X_2(z)$ with ROC= R_2Then,

$x_1(n) * x_2(n) \overset{z}{\leftrightarrow} X_1(z).X_2(z)$, with ROC containing $R_1 \cap R_{2.}$

Proof:

$Z\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{\infty}\{x_1(n) * x_2(n)\}\, z^{-n}$

$$= \sum_{n=-\infty}^{\infty}\{\sum_{m=-\infty}^{\infty} x_1(m)x_2(n-m)\, z^{-n}\}$$

Interchanging the order of summations

$Z\{x_1(n) * x_2(n)\} = \sum_{m=-\infty}^{\infty} x_1\,(m)\,\{\sum_{n=-\infty}^{\infty} x(n-m)z^{-n}\}$

(Since from Time shifting property)

$$=X_2(z)\{\sum_{m=-\infty}^{\infty} x_1\,(m)\, z^{-m}\} = X_1(z).X_2(z)$$

4.6 Accumulation property:

Theorem:

If $x(n) \overset{z}{\leftrightarrow} X(z)$ with ROC = R then $\sum_{k=-\infty}^{n} x(k) \overset{z}{\leftrightarrow} X(z)\,.\frac{1}{1-z^{-1}}$,

With ROC containing R∩ {|z|>1}.

Proof:

$$\sum_{k=-\infty}^{n} x(k) = x(n) * u(n)$$

$$Z\{\sum_{k=-\infty}^{n} x(k)\} = Z\{x(n) * u(n)\}$$

Applying convolution property

$$Z\{\sum_{k=-\infty}^{n} x(k)\} = X(z).\frac{1}{1-z^{-1}}$$

4.7 Time Expansion Property:

Theorem:

If $x(n) \overset{z}{\leftrightarrow} X(z)$ with ROC = R

Then $x_{(m)}(n) \overset{z}{\leftrightarrow} X(z^m)$ with ROC= $R^{1/m}$

That is, if R is $a < |z| < b$, then the new ROC is $a<|z^m|<b$, or $a^{1/m} < |z| < b^{1/m}$.

Also, if $X(z)$ has a pole (or zero) at $z = a$, then (z^m) has a pole (or zero) at $z^{1/m}$.

Proof:

The Z- transform of $x_{(m)}(n)$ is given by

$$Z\{x_{(m)}(n)\} = \sum_{n=-\infty}^{\infty} x_{(m)}(n)z^{-n} = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{m}\right)z^{-n}$$

Changing the variables is performed by letting r = n/m, which also yields r = -∞ as n=-∞ and r=∞as n=∞. Therefore,

$$Z\{x_{(m)}(n)\} = \sum_{r=-\infty}^{\infty} x(r)z^{-mr} = \sum_{r=-\infty}^{\infty} x(r)(z^{m}) - r = X(z^{m})$$

4.8 Differentiation in the Z-Domain:

Theorem:

If $x(n) \overset{z}{\leftrightarrow} X(z)$ with ROC= R

Then, $nx(n) \overset{z}{\leftrightarrow} -z\frac{dX(z)}{dz}$ with ROC= R.

Proof:

Z transform is given by

$$Z\{x(n)\} = X(z)\sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Differentiating above on both sides with respect to z

$$\frac{dX(z)}{dz} = \frac{d}{dz}\{\sum_{n=-\infty}^{\infty} x(n)\,z^{-n}\} = \sum_{n=-\infty}^{\infty} x(n)\frac{d}{dz}\{z^{-n}\} = \sum_{n=-\infty}^{\infty} -nx(n)\,z^{-n-1}$$

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} nx(n)\,z^{-n}$$

Comparing both equations $-z\frac{dX(z)}{dz}$ is the z transform of $nx(n)$ ROC remains the same R because differentiating $X(z)$ will increase the order of the poles present at the same location as earlier.

4.9 Damping Rule:

Theorem:

If $Z(u_n)$=$U(z)$ then $Z(u_n a^{-n})$=$U(az)$.

Proof:

By Definition

$$Z(u_n a^{-n}) = \sum_{n=0}^{\infty} u_n a^{-n} z^{-n}$$

$$= \sum_{n=0}^{\infty} u_n (az)^{-n}$$

$$= U(az)$$

5. TWO BASIC THEOREM

5.1 Initial value theorem

Statement:

If $x(n)$=0, for n < 0 then initial value of $x(n)$.

i.e. $x(0) = \lim_{z\to\infty} X(z)$.

Proof:

We know that $Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$

Expanding the summation

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots\dots\dots$$

Applying the $\lim\limits_{z\to\infty}$ on both sides

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \{ x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots \}$$

i.e.

$$\lim_{z\to\infty} X(z) = x(0)$$

5.1.1 EXAMPLE: Find the initial value of the signal

$$x(n) = 7(\tfrac{1}{3})^n\ u(n) - 6\ (\tfrac{1}{2})^n\ u(n)$$

Solution:

Given signal $x(n) = 7(\frac{1}{3})^n\ u(n) - 6\ (\frac{1}{2})^n\ u(n)$

Applying z-transform

$$X(z) = \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}} = \frac{1-\frac{3}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}$$

Applying initial value theorem

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{1-\frac{3}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})} = 1$$

5.2 Final value theorem

Statement:

If $x(n)$ is causal and $X(z)$ is the Z-transform of $x(n)$and if all the poles of $X(z)$ lie strictly inside the unit circle except possibly for a first order pole at z=1 then,

$$\lim_{N\to\infty} x(n) = \lim_{z\to 1}(1 - z^{-1})X(z)$$

Proof:

Consider the Z-transform of $x(n)$−$x(n-1)$

$$x(n) - x(n-1) \leftrightarrow (1 - z^{-1})X(z)$$

$$Z\{x(n) - x(n-1)\} = \sum_{n=0}^{\infty} x(\text{n}) - x(\text{n}-1)\}\text{z}^{-\text{n}} = (1 - z^{-1})X(z)$$

Also, the above can be written as

$$\lim_{N\to\infty} \sum_{n=0}^{N}\{x(n) - x(n-1)\}\text{z}^{-\text{n}} = (1 - z^{-1})X(z)$$

Applying the limit z→1 on both sides

$$\lim_{z\to1}\{\lim_{N\to\infty}\sum_{n=0}^{N}\{x(n)-x(n-1)\}z^{-n}\} = \lim_{z\to1}\ (1-z^{-1})X(z)$$

LHS after applying the limit z→1 becomes

$$\{\lim_{N\to\infty}\sum_{n=0}^{N}\{x(n)-X(n-1)\}\} = \lim_{N\to\infty}\{\{x(0)-x(-1)+x(1)-x(0)+$$
$$x(2)-x(1)+\ldots+x(N-1)-x(N-2)+x(N)-x(N-1)$$

All terms cancel except $x(n)$, Therefore,

$$\lim_{N\to\infty} x(n) = \lim_{z\to1}(1-z^{-1})X(z)$$

5.2.1 EXAMPLE: Apply the final value theorem to determine

$x(\infty)$ for the signal

$$x(n)=\begin{cases}1, & if\ n\ is\ even\\ 0, & otherwise\end{cases}$$

Solution:

Given that

$$x(n)=\begin{cases}1, & if\ n\ is\ even\\ 0, & otherwise\end{cases}$$

From the definition of the unilateral z-transform, we have

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = \sum_{\substack{n=0\\ n\ is\ even}}^{\infty} (1)z^{-n}$$

Substituting $n = 2r$,

$$X(z) = \sum_{r=0}^{\infty} z^{-2r} = \sum_{r=0}^{\infty}(z^{-2})^{r} = \frac{1}{1-z^{-2}};\ \text{ROC } |z^{-2}| < 1 \to |z| > 1$$

From the final value theorem, we have

$$x(\infty) = \lim_{z \to 1}(1 - z^{-1})x(z)$$

$$= \lim_{z \to 1}(1 - z^{-1})\frac{1}{(1-z^{-2})}$$

$$= \frac{1}{1+z^{-1}}$$

$$= \frac{1}{2}$$

6. EXAMPLE OF Z-TRANSFORM

Example 1:

Determine the Z-transform and ROC of the signal

Solution:

$$Z\{x(n)\} = Z\{u(n)\} - Z\{u(n-10)\}$$

$$= \frac{1}{1-z^{-1}} - \frac{z^{-10}}{1-z^{-1}}$$

$$= \frac{1}{z^{10}}\left\{\frac{z^{10}-1}{z-1}\right\}$$

ROC is the entire Z-plane except $z=0$

Example 2:

Find the Z-transform and ROC of $x(n) = \delta(n+1) - 2\delta(n) + \delta(n-1)$

Solution:

Given signal $x(n) = \delta(n+1) - 2\delta(n) + \delta(n-1)$

Applying Z-transform on both sides

$$X(z) = z - 2 + z^{-1}$$

ROC is entire Z-plane except $z=0$ and $z=\infty$

Example 3:

Find the Z-transform and plot the ROC of

$x(n)$=7 $(\frac{1}{3})^n$ $u(n) - 6\left(\frac{1}{2}\right)^n$ $u(n)$

Solution:

Given signal is $x(n)$=7 $(\frac{1}{3})^n$ $u(n) - 6\left(\frac{1}{2}\right)^n$ $u(n)$ right sided

We know that $b^n u(n) \overset{z}{\leftrightarrow} \frac{1}{1-bz^{-1}}; |z| > b$

Therefore,

$(1/3)^n \overset{z}{\leftrightarrow} \frac{1}{1-(\frac{1}{3})z^{-1}}$;ROC: $|z|>(\frac{1}{3})${Shown in figure a} and

$(1/2)^n \overset{z}{\leftrightarrow} \frac{1}{1-(\frac{1}{2})z^{-1}}$;ROC: $|z|>(\frac{1}{2})${As Shown in figure b}

$$X(Z) = 7\frac{1}{1-(\frac{1}{3})z^{-1}} - 6\frac{1}{1-(\frac{1}{2})z^{-1}}$$

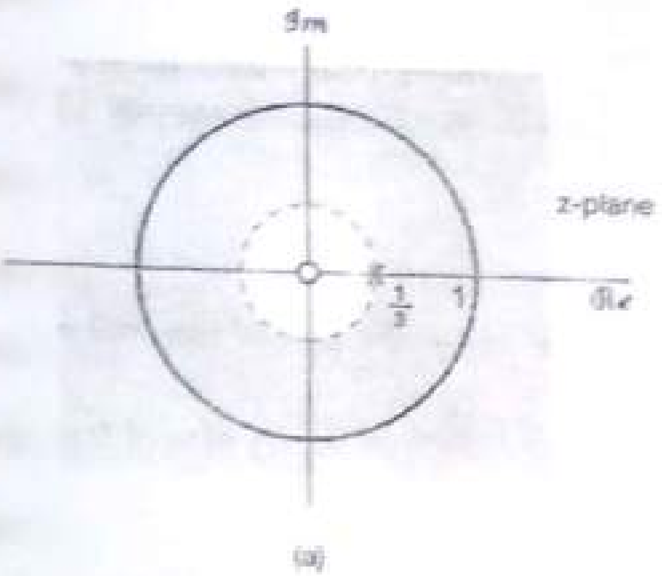
$$= \frac{1-\frac{3}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{3}z^{-1})}$$

$$= \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}$$

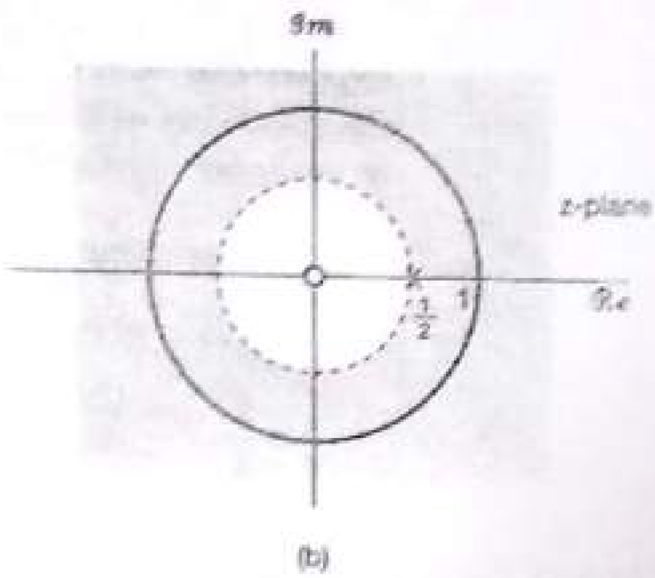
For convergence of $X(z)$, both sums must converge, which requires that the ROC should be

An intersection of $|z| > (\frac{1}{3})$ and $|z| > (\frac{1}{2})$. i.e., $|z| > (\frac{1}{2})$ {shown in figure c}

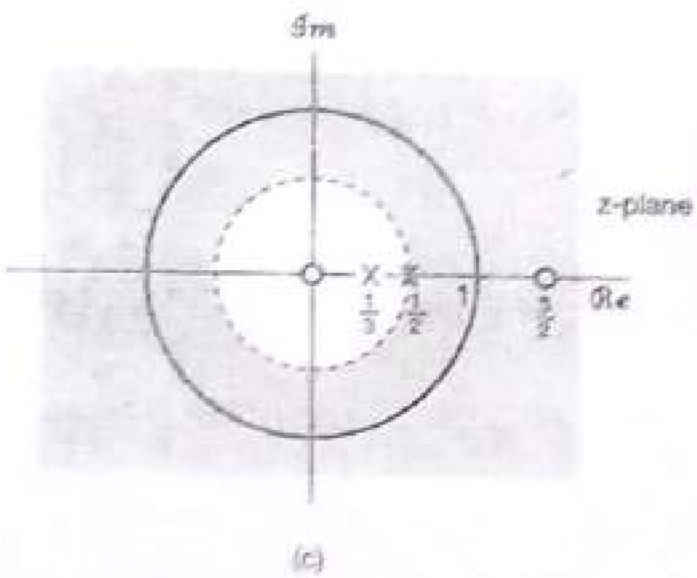
The pole zero plot and ROC are shown in the figure below



(a)



(b)



(c)

Example 4:

Use the convolution property to show that

$u(n) * u(n-1) = nu(n).$

Solution:

Let $x(n) = u(n) * u(n-1) = nu(n)$

Taking the z-transform of $x(n)$ and using the convolution property, we get

$Z\{x(n)\} = Z\{u(n) * u(n-1)\} = (\frac{1}{1-z^{-1}})(\frac{z^{-1}}{1-z^{-1}})$

$$X(z) = \frac{z^{-1}}{(1-z^{-1})^2}$$

Also from differentiation in z-domain property

$$n.u(n) \overset{z}{\leftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right) = \frac{z^{-1}}{(1-z^{-1})^2}$$

Hence $u(n) * u(n-1) = nu(n)$

Example 5:

Apply the final value theorem to determine $x(\infty)$ for the signal

$$x(n) = \begin{cases} 1, & if\ n\ is\ even \\ 0, & otherwise \end{cases}$$

Solution:

Given that

$$x(n) = \begin{cases} 1, & if\ n\ is\ even \\ 0, & otherwise \end{cases}$$

From the definition of the unilateral Z-transform, we have

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = \sum_{\substack{n=0 \\ n\ is\ even}}^{\infty} (1) z^{-n}$$

Substituting n=2r,

$X(z) = \sum_{r=0}^{\infty} z^{-2r} = \sum_{r=0}^{\infty} (z^{-2})^r = \frac{1}{1-z^{-2}}$; ROC $|z^{-2}| < 1 \rightarrow |z| > 1$

From the final value theorem, we have

$x(\infty) = \lim_{z\rightarrow 1}(1 - z^{-1})x(z)$

$= \lim_{z\rightarrow 1}(1 - z^{-1}) \frac{1}{(1-z^{-2})}$

$= \frac{1}{(1-z^{-1})}$

$= \frac{1}{2}$

7. SOME STANDARD RESULTS

7.1 The application of the damping rule leads to the following standard results.

1. $Z(na^n) = \frac{az}{(z-a)^2}$

We know that $Z(n) = \frac{z}{(z-1)^2}$

Applying Damping rule, we have

$$Z(na^n) = U(a^{-1}z)$$

$$= \frac{a^{-1}}{(a^{-1}z-1)^2}$$

$$= \frac{az}{(z-a)^2}$$

2. $Z(n^2a^2) = \frac{az^2+a^2z}{(z-a)^3}$

We know that $Z(n^2) = \frac{z^2+z}{(z-1)^3}$

Applying Damping rule, we have

$$Z(n^2a^2) = U(a^{-1}z) = \frac{(a^{-1}z)^2+a^{-1}z}{(a^{-1}z-1)^3}$$

$$= \frac{a(z^2+az)}{(z-1)^3}$$

3. $\boldsymbol{Z(\cos n\theta)} = \frac{z(z-\cos\theta)}{z^2-2az\cos\theta+a^2}$

We know that $z(1)= \frac{z}{z-1}$

Applying Damping rule, we have

$$Z(e^{-in\theta}) = Z\left(e^{-i\theta}\right)^n.1$$

$$= \frac{ze^{i\theta}}{ze^{i\theta}-1}$$

$$= \frac{z}{z-e^{-i\theta}}$$

$$= \frac{z(z-e^{i\theta})}{(z-e^{-i\theta})(z-e^{i\theta})}$$

$$= \frac{z(z-\cos\theta)-iz\sin\theta}{z^2-z\left(e^{i\theta}+e^{-i\theta}\right)+1}$$

$$= \frac{z(z-\cos\theta)-iz\sin\theta}{z^2-2z\cos\theta+1}$$

$$= \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} - i\,\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

Now, equating real part, we get

$$\mathrm{Z}(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

4. $Z(sinn\theta)=\frac{z\sin\theta}{z^2-2z\cos\theta+1}$

We know that, $Z(1)=\frac{z}{z-1}$

Applying Damping rule, we have

$$Z(e^{-in\theta}) = Z\left(e^{-i\theta}\right)^n.1$$

$$= \frac{ze^{i\theta}}{ze^{i\theta}-1}$$

$$= \frac{z}{z-e^{-i\theta}}$$

$$= \frac{z(z-e^{i\theta})}{(z-e^{-i\theta})(z-e^{i\theta})}$$

$$= \frac{z(z-\cos\theta)-iz\sin\theta}{z^2-z\left(e^{i\theta}+e^{-i\theta}\right)+1}$$

$$= \frac{z(z-\cos\theta)-iz\sin\theta}{z^2-2z\cos\theta+1}$$

$$= \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} - i\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

Now, equating imaginary part, we get

$$Z(\sin n\theta)=\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

5. $Z(a^n \cos n\theta) = \frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$

We know that, $Z(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$

By damping rule, we have

$$Z(a^n \cos n\theta) = \frac{a^{-1}z(a^{-1}z-\cos\theta)}{(a^{-1}z)^2-2(a^{-1}z)\cos\theta+1}$$

$$= \frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$$

6. $Z(a^n \sin n\theta) = \frac{az\sin\theta}{z^2-2az\cos\theta+a^2}$

We know that $Z(\sin n\theta) = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$

By damping rule, we have

$$Z(a^n \sin n\theta) = \frac{(a^{-1}z)\sin\theta}{(a^{-1}z)^2-2(a^{-1}z)\cos\theta+1}$$

$$= \frac{az\sin\theta}{z^2-2az\cos\theta+a^2}$$

7.2 SOME USEFUL Z-TRANSFORM

Sr.No	Sequence $u_n (n \geq 0)$	Z-transform U(z) = Z(u_n)
1.	K	Kz/(z-1)
2.	-k	Kz/(z+1)
3.	N	$z/(z-1)^2$
4.	n^2	$(z^2+z)(z-1)^2$
5.	n^p	-zd/dz[Z(n^{p-1})], $p + ve\ integer$
6.	a^n	z/(z-a)
7.	na^n	az/$(\boldsymbol{z}-\boldsymbol{a})^2$
8.	n^2a^n	$(az^2+za^2)/(z-a)^3$
9.	$\sin n\theta$	$\frac{z \sin\theta}{z^2 - 2z\cos\theta + 1}$
10.	$\cos n\theta$	$\frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$
11.	$a^n \sin n\theta$	$\frac{az \sin\theta}{z^2 - 2az\cos\theta + a^2}$
12.	$a^n \cos n\theta$	$\frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$
13.	$\sinh n\theta$	$\frac{z \sinh\theta}{z^2 - 2z\cos h\theta + 1}$
14.	$\cosh n\theta$	$\frac{z(z - \cosh\theta)}{z^2 - 2z\cos h\theta + 1}$
15.	$a^n \sinh n\theta$	$\frac{az \sinh\theta}{z^2 - 2az\cosh\theta + a^2}$
16.	$a^n\ \mathrm{cohs}\, n\theta$	$\frac{z(z - a\cos h\theta)}{z^2 - 2az\cosh\theta + a^2}$
17.	$a^n u_n$	U(z/a)
18.	u_{n+1}	Z[U(z)-u_o]
19.	u_{n+2}	$z^2[U(z) - u_0 - u_1 z^{-1}]$
20.	u_{n+3}	$z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$
21.	u_{n-k}	$z^{-k}[U(z)]$
22.	nu_n	-zd/dz[U(z)]
23.	u_0	$\lim_{z\to\infty} U(z)$
24.	$\lim_{n\to\infty} u_n$	$\lim_{z\to\infty}[(z-1)U(z)]$

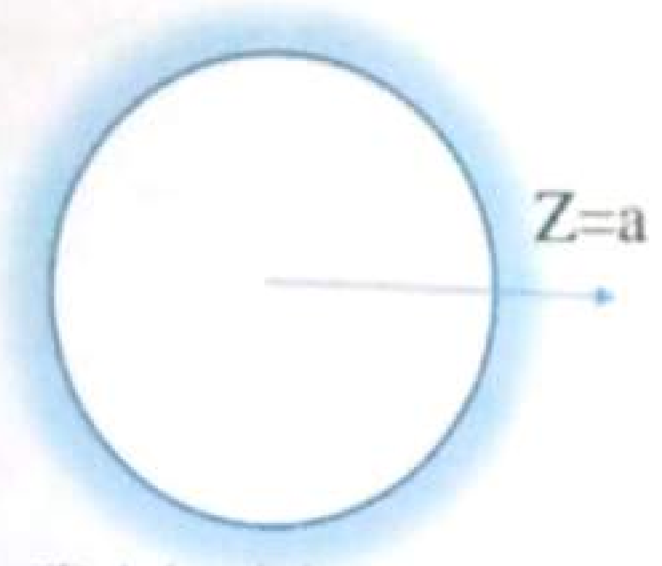
8. TWO SIDED Z-TRANSFORM

Two sided Z-transform is given as:

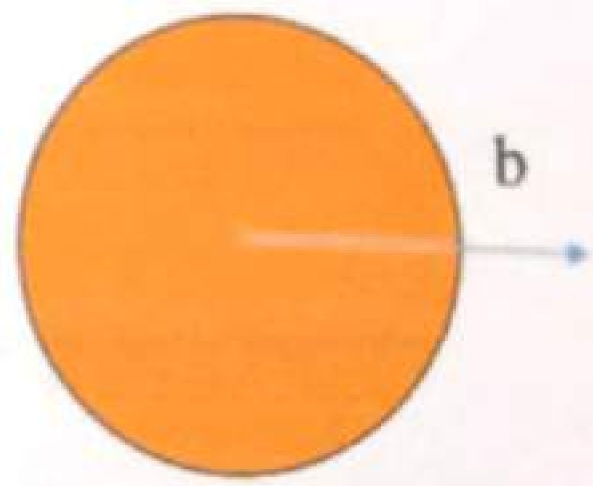
$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \qquad \text{...... (1)}$$

In this case, the sequence is two-sided and the region of convergence for (1) is the annular region $|b| < |z| < |c|$

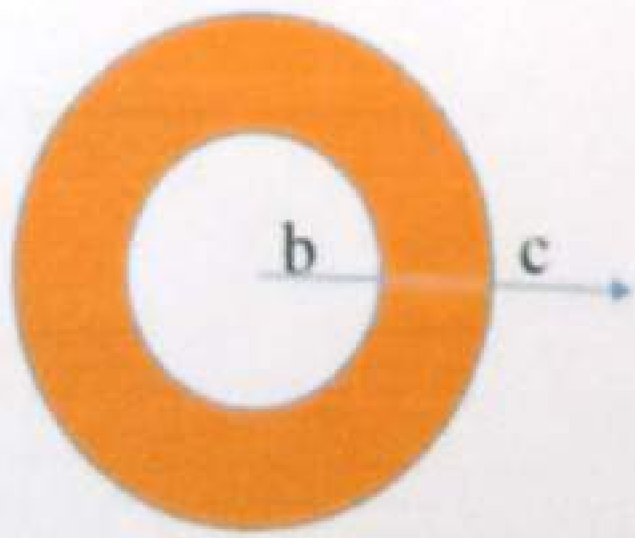
The linear circle bounds the terms in negative powers of z and the outer circle bounds the terms in positive powers of z. The shaded annulus of convergence is necessary for the two sided sequences and its Z-transforms to exist.



(i) $|z| > |a|$



(ii) $|z| < |b|$



(iii) $|b| < |z| < |c|$

EXAMPLES:

8.1 Find the Z-transform and region of convergence of

$$u(n) = \begin{cases} 4^n \; for \; n < 0 \\ 2^n \; for \; n \leq 0 \end{cases}$$

solution: By definition

$$Z[u(n)] = \sum_{-\infty}^{\infty} u(n)\, z^{-n}$$

$$=\sum_{-\infty}^{-1} 4^n z^{-n} + \sum_{n=0}^{\infty} 2^n z^{-n}$$

Putting –n=m in the first series, we get

$$Z[u(n)]= \sum_{1}^{\infty} 4^{-m} z^{m} + \sum_{n=0}^{\infty} 2^n z^{-n}$$

$$= \left\{\frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + \cdots\right\} + \{1 + \frac{2}{z} + \frac{2^2}{z^2} + \cdots\}$$

$$= \frac{1}{4}\left\{1 + \frac{z}{4} + \left(\frac{z}{4}\right)^2 + \cdots\right\} + \{1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \cdots\}$$

$$= \frac{z}{4}\frac{1}{1-\left(\frac{z}{4}\right)} + \frac{1}{1-\left(\frac{2}{z}\right)}$$

$$= \frac{z}{4-z} + \frac{z}{2-z}$$

$$= \frac{2z}{(4-z)(z-2)}$$

Now the two series in (i) being G.P. Will be convergent if $|\frac{z}{4}|<1$ and $|\frac{2}{z}|<1$ i.e., if $|Z|<4$ and $2<|Z|$. i.e. $2<|Z|<4$.

Hence $Z[u(n)]$ is convergent if Z lies between the annulus as shown shaded in above fig. Hence ROC is $2<|Z|<4$.

8.2 $u(n)=\binom{n}{k}, n\geq k$

solution: By definition

$$Z[u(n)]=\sum_{-\infty}^{\infty}\binom{n}{k}z^{-n}$$

$$=\sum_{n=k}^{\infty}\binom{n}{k}2^{n}z^{-n}$$

Two find the sum of this series, we replace n by k+r.

$$Z[u(n)]=\sum_{r=0}^{\infty}\binom{k+r}{k}z^{-(k+r)}$$

$$= z^{-k}\sum_{r=0}^{\infty}\binom{k+r}{r}z^{-r}$$

$$= z^{-k}[1+\binom{k+1}{1}z^{-1}+\binom{k+2}{2}+\cdots]$$

$$= z^{-k}(1-\frac{1}{z})^{-(k+1)}$$

This series is convergence for $|\frac{1}{z}|<1$ i.e., for $|z|>1$.

Hence ROC is $|z|>1$.

8.3 $f(n) = 2^n, n<0$

solution:

Assuming that $f(n) = 0$ for $n \geq 0$ we have

$$Z[f(n)] = \sum_{-\infty}^{\infty} f(n)\, z^{-n}$$

$$= \sum_{-\infty}^{-1} 2^n\, z^{-n}$$

$$= \sum_{m=1}^{\infty} 2^{-m}\, z^m \quad \text{Where } m = -n$$

$$= \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots \infty$$

$$= \frac{z}{2}\{1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots \infty\}$$

$$= \frac{z}{2} \cdot \frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= \frac{z}{2-z}$$

This series being a G.P. is convergent if $\left|\frac{z}{2}\right| < 1$ i.e. $|z| < 2$.

Hence ROC is $|z| < 2$.

8.4. $f(n) = 5^n / n!\ , n \geq 0$.

solution: By definition

$$z[u(n)] = \sum_{n=0}^{\infty} \frac{5^n}{n!} \cdot z^{-n}$$

$$= \sum_{0}^{\infty} \frac{\left(\frac{5}{z}\right)^n}{n!}$$

$$= 1 + \left(\frac{5}{z}\right) + \frac{1}{2!}\left(\frac{5}{z}\right)^2 + \frac{1}{3!}\left(\frac{5}{z}\right)^3 + \cdots \infty$$

$$= e^{\frac{5}{z}}$$

The above series is convergent for all value of z.

Hence ROC is the entire z-plane.

9. INVERSE Z-TRANSFORM

We can obtain the inverse Z-transforms using any of the following three methods.

9.1 Power series method:

This is the simplest of all the methods of finding the inverse Z-transform. If $U(z)$ is expressed as the ratio of two polynomials which cannot be factorized, we simply divide the numerator by the denominator and take the inverse Z-transform of each term in quotient.

EXAMPLE:

1. Find the inverse Z-transform of $\log(\frac{z}{z+1})$ by power series method.

Solution:

Putting z= $\frac{1}{y}$

$$U(z) = \log(\frac{\frac{1}{y}}{\frac{1}{y}+1})$$

$$= -\log(1+y)$$

$$= -y + \frac{1}{2}y^2 - \frac{1}{3}y^3 + \cdots$$

$$= -z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{3}z^{-3} + \cdots$$

Thus, $u_n = \begin{cases} 0 \; for \; n = 0 \\ \frac{(-1)^n}{n} \; otherwise \end{cases}$

2. Find the inverse Z-transform of $\frac{z}{(z+1)^2}$ by division method.

Solution:

$$U(z)=\frac{z}{z^2+2z+1}$$

$$=z^{-1}-\frac{2+z^{-1}}{z^2+2z+1}$$

$$=z^{-1}-2z^{-2}+\frac{3z^{-1}+2z^{-2}}{z^2+2z+1}$$

$$=z^{-1}-2z^{-2}+3z^{-3}-\frac{4z^{-2}+3z^{-3}}{z^2+2z+1}$$

Continuing this process of division, we get an infinite series i.e.

$$U(z)=\sum_{n=0}^{\infty}(-1)^{n-1}nz^{-n}$$

Thus,

$$u_n=(-1)^{n-1}n$$

9.2 Partial fraction method:

This method is similar to that of finding the inverse Laplace transforms using partial fractions. The method consists of decomposing $\frac{U(z)}{z}$ into partial fractions, multiplying the resulting expansion by z and then inverting the same.

EXAMPLE:

1. Find the inverse Z-transform of $\frac{2z^2+3z}{(z+2)(z-4)}$

Solution:

We write $U(z)= \frac{2z^2+3z}{(z+2)(z-4)}$

$$\frac{U(z)}{z} = \frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4}$$

$$\therefore U(z) = \frac{1}{6}\frac{z}{z+2} + \frac{11}{6}\frac{z}{z-4}$$

On inversion, we have

$$u_n = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n$$

2. Find the inverse Z-transform of $\frac{(z)^3-20z}{(z-2)^3(z-4)}$

Solution:

We write $U(z)= \frac{(z)^3-20z}{(z-2)^3(z-4)}$

$$\frac{U(z)}{z} = \frac{z^2-20z}{(z-2)^3(z-4)} = \frac{A+Bz+cz^2}{(z-2)^3} + \frac{D}{z-4}$$

Readily we get D = $\frac{1}{2}$

Multiplying throughout by $(z-2)^3(z-4), we\ get$

$$z^2 - 20 = (A + Bz + cz^2)(z-4) + D(z-2)^3$$

Putting z=0,1,−1 successively and solving the resulting simultaneous equations, we get A=6, B=0, C=$\frac{1}{2}$

Thus, $U(z) = \frac{1}{2}\frac{12z+z^3}{(z-2)^3} - \frac{z}{(z-4)}$

$$= \frac{1}{2}\frac{z(z-2)^2+4z^2+8z}{(z-2)^3} - \frac{z}{(z-4)}$$

$$= \frac{1}{2}\left\{\frac{z}{(z-2)} + 2\frac{2z^2+4z}{(z-2)^3}\right\} - \frac{z}{(z-4)}$$

On inversion, we get

$$u_n = \frac{1}{2}(2^n + 2n^2 2^n) - 4^n$$

$$= (2)^{n-1} + n^2 2^n - 4^n$$

3. Find the inverse Z-transform of $\frac{2(z^2-5z+6.5)}{[(z-2)(z-3)^2]}, for\ 2<|z|<3.$

Solution:

Splitting into partial fraction, we obtain

$$U(z)= \frac{2(z^2-5z+6.5)}{[(z-2)(z-3)^2]}$$

$$= \frac{A}{(z-2)} + \frac{B}{(z-3)} + \frac{C}{(z-3)^2}$$

$$U(z) = \frac{1}{(z-2)} + \frac{1}{(z-3)} + \frac{1}{(z-3)^2}$$

$$= \frac{1}{2}\ (1-\frac{2}{z})^{-1} - \frac{1}{3}(1-\frac{z}{3})^{-1} + \frac{1}{9}(1-\frac{z}{3})^{-1}$$

$$= \frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right) - \frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\cdots\right)$$

$$+\frac{1}{9}\left(1+\frac{2z}{3}+\frac{3z^2}{9}+\frac{4z^3}{27}+\cdots\right)$$

Where, 2<|z|<3

$$=\left(\frac{1}{2}+\frac{2}{z^2}+\frac{2}{z^3}+\cdots\right)-\left(\frac{1}{3}+\frac{z}{3^2}+\frac{z}{3^3}+\cdots\right)$$

$$+\left(\frac{1}{3^2}+\frac{2z}{3^3}+\frac{2z^2}{3^3}+\cdots\right)$$

$$=\sum_{n=1}^{\infty} 2^{n-1}\, z^{-n}-\sum_{n=0}^{\infty}(\tfrac{1}{3})^{n+1}+\sum_{n=0}^{\infty}(n+1)\,(\tfrac{1}{3})^{n+2} z^n$$

On inversion, we get $u_n = 2^{n-1},\ n \geq 1 \quad and$

u_n=$-(n+2)3^{n-2},\ \ n \leq 0$

9.3 Inversion integral method:

The inverse Z-transform of U(z) is given by the formula

$$u_n = \frac{1}{2\pi i}\int U(z)z^{n-1}\, dz$$

= sum of residue of $U(z)z^{n-1}$ at the poles of $U(z)$ which are inside the contour C drawn according to the ROC given

The following example will illustrate of this formula:

EXAMPLE:

1. using the inversion integral method, find the inverse Z-transformation of $\frac{z}{(z-1)(z-2)}$.

Solution:

Let $U(z)=\frac{z}{(z-1)(z-2)}$

Its poles are at z=1 and z=2

Using $U(z)$in the inversion integral, we have

$$u_n = \frac{1}{2\pi i}\int U(z)z^{n-1}\,dz$$

= sum of residue of $U(z)z^{n-1}$at z=1 and z=2.

Now,

$$Res[U(z)z^{n-1}]_{(z=1)} = \lim_{z\to 1}(z-1)\frac{z^n}{(z-1)(z-2)}$$

$$= -1$$

And

$$Res[U(z)z^{n-1}]_{(z=2)} = \lim_{z\to 2}(z-2)\frac{z^n}{(z-1)(z-2)} = 2^n$$

Thus, the required inverse Z-transform

$$u_n = 2^n - 1, n = 0,1,2, \dots$$

2. Find the inverse Z-transform of $\frac{2z}{(z-1)(z^2+1)}$.

Solution:

Let $U(z)= \frac{2z}{(z-1)(z+i)(z-i)}$

It has three poles at $z=1, z=\pm i$

Using, $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i}\int U(z) z^{n-1}\, dz$$

= sum of residue of $U(z)z^{n-1}$ at $z=1$ and $z=\pm i$.

Now,

$$Res[U(z)z^{n-1}]_{(z=1)} = \lim_{z\to 1}(z-1)\frac{2z^n}{(z-1)(z^2+1)}$$

$$= 1$$

$$Res[U(z)z^{n-1}]_{(z=i)} = \lim_{z\to i}(z-i).\frac{2z^n}{(z-1)(z+i)(z-i)}$$

$$= \frac{-(i)^n}{1+i}$$

$$Res[U(z)z^{n-1}]_{(z=-i)} = \lim_{z\to -i}(z+i).\frac{2z^n}{(z-1)(z+i)(z-i)}$$

$$= \frac{(-i)^n}{i-1}$$

Hence, $\quad u_n = 1 - \frac{(i)^n}{1+i} - \frac{(-i)^n}{i-1}$

10. APPLICATION

Z-transform is used to covert discrete time domain into a complex frequency domain where, discrete time domain represents an order of complex or real numbers. It is generalize form of Fourier transform, which we get when generalize Fourier transform and get Z-transform. The reason behind this is that Fourier transform is not sufficient to converge on all sequence and when we do this thing then we get the power of complex variable theory that we deal with non-contiguous time system and signals.

This transform is used in many applications of mathematics and signal processing. The lists of applications of Z-transform are as under:

- Uses to analysis of digital filters.
- Used to simulate the continuous systems.
- Analyze the linear discrete system.
- Used to finding frequency response.
- Analysis of discrete signal.
- Helps in system design and analysis and also checks the systems stability.
- For automatic controls telecommunication.
- Enhance the electrical and mechanical energy to provide dynamic nature of system.

Z-transforms represent the system according to their location of poles and zeros of the system during transfer function that happens only in complex plane. It is closely related to Laplace Transform. Main functionality of this transform to provides access to transient behavior (transient behavior means changeable) that Monitors all states stability of a system or all behavior either static or dynamic. This transform is generalize form of Fourier transform from a discrete time signals and Laplace transform is also a generalize form of Fourier transform but from continuous time signals.

- **Application to difference equation:**

Just as the Laplace transforms method is quite effective for solving linear differential equations the Z-transforms are quite useful for solving linear difference equations.

The performance of discrete systems is expressed by suitable difference equations. Also Z-transform plays an important role in the analysis and representation of discrete-time systems, the solution of difference equations is required for Which Z-transform method proves useful.

Working procedure to solve a linear difference equation with constant coefficient by Z-transform:

- Take the Z-transform of both sides of the difference equations using the formulae of application to difference equations and the given conditions.
- Transpose all terms without $U(z)$to the right.
- Divide by the coefficient of $U(z)$, getting $U(z)$ as a function of z.
- Express this function in terms of the Z-transform of Known functions and take the inverse Z-transform of both sides. This gives un as a function of n which is the desired solution.

EXAMPLE:

1. Using the Z-transform, solve

$u_{n+2} + 4u_{n+1} + 3u_n = 3^n$ ***with*** $u_0 = 0, u_n = 1$.

Solution:

$z(u_n) = U(z)$, Then $z(u_{n+1}) = z[U(z) - u_0]$

$z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$

Taking the Z-transform of both side, we get

$$z^2[U(z) - u_0 - u_1 z^{-1}] + 4z[U(z) - u_0] + 3U(z)$$

$$= z/(z-3)$$

Using the given conditions, it reduces to

$U(z)(z^2 + 4z + 3) = z + z/(z-3)$

$$\frac{U(z)}{z} = \frac{1}{(z+1)(z+3)} + \frac{1}{(z-3)(z+1)(z+3)}$$

$$= \frac{3}{8}\frac{1}{(z+1)} + \frac{1}{24}\frac{1}{(z-3)} - \frac{5}{12}\frac{1}{(z+3)}$$

On breaking into partial fractions.

$$U(z) = \frac{3}{8}\frac{z}{(z+1)} + \frac{1}{24}\frac{z}{(z-3)} - \frac{5}{12}\frac{z}{(z+3)}$$

On inversion, we obtain

$$u_n = \frac{3}{8}z^{-1}\left(\frac{z}{z+1}\right) + \frac{1}{24}z^{-1}\left(\frac{1}{z-3}\right) - \frac{5}{12}z^{-1}\left(\frac{1}{z+3}\right)$$

$$= \frac{3}{8}(-1)^n + \frac{1}{24}(3)^n - \frac{5}{12}(-3)^n$$

❖REFERENCE :

1. B.S.GREWAL, Khanna Publisher, "Higher engineering mathematics".
2. ELIAHU Ibrahim JURY,Huntington,N.Y.,R.E. Kriger Publisher, "Theory and application of Z-transform",1973.
3. REFAAT El ATTAR, Lulu.Com Publisher, "Lecture Notes on Z-transform", 30 July, 2006.

A

PROJECT REPORT

ON

FOURIER SERIES

Is fulfillment of requirements for the degree of

BACHELOR OF SCIENCE

IN

MATHEMATICS

DEPARTMENT OF MATHEMATICS

M P SHAH ARTS AND SCIENCE COLLEGE,
SURENDRANAGAR

2018-19

A FINAL YEAR PROJECT REPORT ON

"FOURIER SERIES"

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This is Certify that GROUP E. a student of class T.Y.B Sc Mathematics has successfully completed her Mathematics project on **"FOURIER SERIES"**. This Project carried out department of Methematic M.P. Shah Arts & Science college, Saurendranagar, comprises the result of independent & original work for the partial fulfillment of the degree of Bachelor of science in Mathematic.

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principal

AKNOWLEDGEMENT

I would like to express my special thanks of gratitude to my Professor Dr J.R.JOSHI who give me golden opportunity to do this wonderful project on the topic FOURIER SERIES , which also helped me in doing a Lots of Research and I came to know about so many new Things I am really thankful to them.

Secondly I would also like to thanks my parents and friends Who helped me a lot in finalizing this project within the Limited time frame.

DECLARATION

I hereby declare that this project report entitled **FOURIER SERIES** has been prepared by me is an original work

Submitted to Saurashtra University toward partial fulfilment

Of the requirement of the award Bachelor of science.

I also hereby declare that this project report has not been

Submitted at any time to any other University or institute for the award of any Degree.

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1. INTRODUCTION

Jean Baptiste Joseph Fourier (1768-1830)

It was around 1804 that Fourier did his important mathematical work on important memory on the propagation of heat in solid bodies, which was read to the Paris institute memory on December 21,1807 and a committee consisting of Lagrange, Laplace, Mange and Lacroix was set up to report on the work.

The Institute set as a prize competition subject the propagation of heat in solid bodies for the 1811 mathematics prize. Fourier submitted his 1807 memory together with addition work on the cooling of infinite solids and terrestrial and radiant heat. Only one other entry was received and the committee set up to decide on the award of the prize, Lagrange , Laplace , Malus, Haul and Legendre ,awarded Fourier the award of the prize. The report was not however completely favorable and states:

"…..the manner in which the author arrives at these equations is not exempt of difficulties and that his analysis to integrate them still leaves something to be desired on the score of generality and even rigor."

1.1: HISTORY

Fourier series is simply decomposing periodic function to summation of simple sine & cosine to immediate the graph of periodic function. It may not be perfectly same as smooth periodic function such as multi-conditional function. However, it accordingly represents periodic function with summation notation that we have worked on often in mathematical progression.

It begins as study of trigonometric series by famous &Bernoulli later Jean –Baptiste Fourier contributed most to creating Fourier series as for solving heat equation in metal plate.

2. Periodic Functions

2.1 Definition:-

A function f(x) is said to be periodic p if f (x+p) = f(x) for all x.

Where, P>0 is called periodic, or more specifically. The number P is called a period of f. If f is non constant, we define the fundamental period, or simply, the period of f to be the smallest positive number P.

2.2 Examples

Example 1:-

$$f(x) = \cos x$$

Solution: $f(x+2\pi) = \cos(x + 2\pi)$

$= \cos x\ \cos 2\pi - \sin x\ \sin 2\pi$

$= \cos x$

$= f(x)$

Hence $\cos x$ is periodic of period 2π .

Example 2:- $f(x) = \sin 4x$

Solution: $f(x+\frac{\pi}{2}) = \sin\left(\left(x + \frac{\pi}{2}\right)\right)$

$= \sin(4x + 2\pi)$

$= \sin 4x \cos 2\pi + \cos 4x\ \sin$

$=f(x)$

Hence $\sin nx$ is periodic of period $\frac{\pi}{2}$, observe that 2π is also a period $of\ \sin 4x$.

❖Useful Identities

- $\sin(ax+b)=\sin ax \cos b + \cos ax \sin b$
- $\cos(ax+b) = \cos ax \cos b - \sin ax \sin b$

<u>Notes</u>:-

- Any function can be considered periodic with period zero; this period is trivial and is not considered as a period.
- If p is a period of f, then np is a period for any integer n.

Proof:

Want: f (x + np) =f(x)

We know that f (x +p) = f(x)

F (x+2p) =f (x +p +p) =f (x+ p) =f(x)

F (x+3p) =f(x+p+2p) =f (x+ p) =f(x)

F (x-p) =f(x-p +p) =f(x)

- If p is a period then $\frac{p}{2}$ is not necessarily a period.

2.3Fundamental Period

The most interesting period for a periodic function is the smallest positive period; this period is called the fundamental period.

The fundamental period of

- $\sin 3x$ is $\frac{2\pi}{3}$
- $\sin x$ is 2π

2.4 Period of Multiple Functions

Definition:-

If f and g are periodic of period p then so is f + g.

Proof:

Denote f + g by h

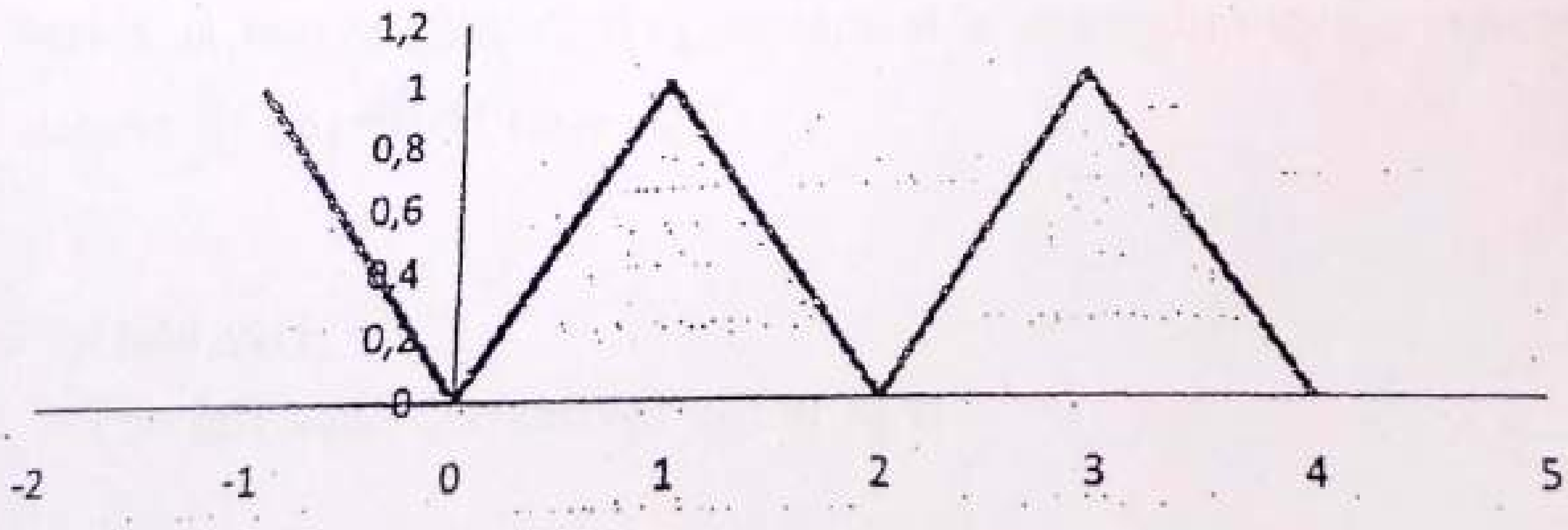
Want h(x + p) = h(x)

h (x + p) = f(x + p + g(x + p))

=f(x) = g(x)

=h(x) h is periodic of period p

If f is periodic of period p then the graph of f repeats itself every p units



P=2

Therefore if we know the curve of a periodic function on $[-\frac{p}{2},\frac{p}{2}]$, then we can draw the entire graph.

2.5 Functions of Any Period p = 2L

In general

$$F\{f\} = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

Where period, p=2L

2.6 : Formal definition:-

If f(x) is periodic, with period 2π, piecewise continuous in $[-\pi,\pi]$, has a left and right-hand derivative at each $x_0 \in[-\pi,\pi]$, then its Fourier series is convergent to f(x), except at a discontinuity x_0, where the sum is $\frac{1}{2}[f(x_0+0) + f(x_0 - 0)]$.

- REMARK

The left hand derivatives of f at x_0 is

$$lim_{h\to 0_-}\frac{f(x_0-0)-f(x_0-h)}{h}$$

Where $f(x_0 - 0) = lim_{x\to x_{0-}} f(x)$ and similarly the right hand derivative of f at x_0 is

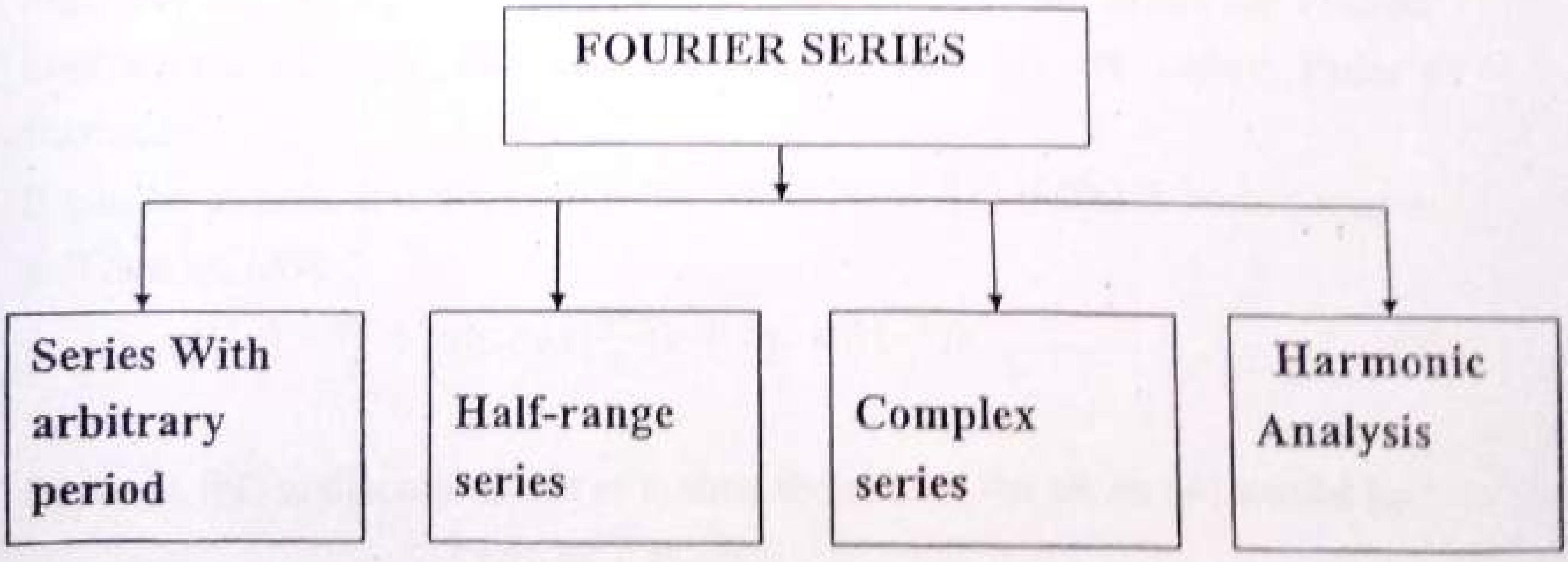
$$lim_{h\to 0+}\frac{f(x_0+h)-f(x_0+0)}{h}$$

We can in these circumstances write

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin n\pi)$$

3. FOURIER SERIES

Fourier series is an infinite series representation of periodic function in terms of the trigonometric sine and cosine functions. Most of the single valued functions which occur in applied mathematics can be expressed in the form of Fourier series, which is in terms of sines and cosines. Fourier series is to be expressed in terms of periodic functions- sines and cosines. Fourier series is a very powerful method to solve ordinary and partial differential equations, particularly with periodic functions appearing as non-homogeneous terms. We know that, Taylor's series expansion is valid only for functions which are continuous and differentiable. Fourier series is possible not only for continuous functions but also for periodic functions, functions which are discontinuous in their values and derivatives. Further, because of the periodic nature, Fourier series constructed for one period is valid for all values



3.2 Particular Cases:

Case (i)

Suppose a=0. Then f(x) is defined over the interval (0,2l). Formula (1),(2),(3) reduce to

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos(\frac{np}{l})xdx \text{ , } n= 1, 2, \ldots\ldots \infty \qquad (6)$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{np}{l}\right)xdx,$$

Then the right-hand side of (5) is the Fourier expansion of f(x) over the interval (0,2l).

If we set l=π , then f(x) is defined over the interval (0, 2π). Formulae (6) reduce to

$$a_0 = \frac{1}{p}\int_0^{2p} f(x)dx$$

$$a_n = \frac{1}{p}\int_0^{2p} f(x)\cos nxdx \text{ , } \qquad n= 1, 2, \ldots\ldots, \infty \qquad (7)$$

$$b_n = \frac{1}{p}\int_0^{2p} f(x)\sin nxdx \qquad n = 1, 2, \ldots\ldots, \infty$$

Also, in this case, (5) becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \qquad (8)$$

Case (ii)

Suppose a=-l. Then f(x) is defined over the interval (-l, l). Formulae (1), (2), (3) reduce to

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx \qquad n = 1, 2, \ldots\ldots\ldots\ldots, \infty \qquad (9)$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{np}{l}\right)xdx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{np}{l}\right)xdx \qquad n=1, 2, \ldots\ldots\ldots\ldots, \infty$$

Then the right-hand side of (5) is the Fourier expansion of f(x) over the interval (-l, l).

If we set l=π, then f(x) is defined over the interval (-π, π).Formulae (9) reduce to

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos nxdx\,, \qquad n=1, 2, \ldots\ldots\ldots\ldots, \infty \qquad (10)$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin nxdx \qquad n=1, 2, \ldots\ldots\ldots\ldots, \infty$$

Putting l = π .in (5) , we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

3.3 Examples:-

Example 1:-

Obtain the Fourier expansion of

$F(x) = \frac{1}{2}(p - x)$ in $-\pi < x < \pi$

Solution: - we have,

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx = \frac{1}{p}\int_{-p}^{p} \frac{1}{2}(p - x)dx$$

$$= \frac{1}{2p}[px - \frac{x^2}{2}]_{-p}^{p} = p$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos nxdx = \frac{1}{p}\int_{-p}^{p} \frac{1}{2}(p - x)\cos nxdx$$

Here we use integration by parts, so that

$$a_n = \frac{1}{2p}[(p - x)\frac{\sin nx}{n} - (-1)\left(\frac{-\cos nx}{n^2}\right)]_{-p}^{p}$$

$$= \frac{1}{2p}[0] = 0$$

$$b_n = \frac{1}{p}\int_{-p}^{p} \frac{1}{2}(p - x)\sin nxdx$$

$$= \frac{1}{2p}[(p - x)\frac{-\cos nx}{n} - (-1)\left(\frac{-\sin nx}{n^2}\right)]_{-p}^{p}$$

$$= \frac{(-1)^n}{n}$$

Using the values of a_0, a_n and b_n in Fourier expansion

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

We get, $f(x) = \frac{p}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$

This is the required Fourier expansion of the given function.

Example 2:-

Find the Fourier series of

$f(x) = x^2, \quad -1 < x < 1$

solution :

Integration by parts

x^2	$\cos n\pi x$
↘	$\frac{\sin(n\pi x)}{n\pi}$
$2x$ ↘	$\frac{-\cos(n\pi x)}{\pi^2 n^2}$
2 ↘	
0 →	$-\int$ $\frac{-\sin(n\pi x)}{\pi^3 n^3}$

In the example, p=2(period =2)

In this case when p=2L

Thus in our example L=1

$$a_0 = \frac{1}{2}\int_{-1}^{1} x^2 dx = \frac{1}{3}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} x^2 \sin n\pi x \, dx = 0$$

$$a_n = \frac{1}{1}\int_{-1}^{1} x^2 \cos n\pi x \, dx$$

$$= \frac{1}{1}\int_{-1}^{1} x^2 \cos n\pi x \, dx = \frac{2x \cos n\pi x}{n^2\pi^2}\Big|_{-1}^{1}$$

$$= \frac{2}{n^2\pi^2}[(-1)^n + (-1)^n]$$

$$= \frac{4(-1)^n}{n^2\pi^2}$$

So, $F\{f\} = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cos(n\pi x)$

Example 3:-

Find the Fourier series of f(x)=x^2, 0<x<1

Solution :-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x\ ,\ l = 1$$

$$a_0 = \int_0^2 x^2 dx$$

$$= \left[\frac{x^2}{3}\right]_0^2 = \frac{8}{3}$$

$$a_n = \int_0^2 x^2 \cos n\pi x dx$$

$$= \left[x^2 \frac{\sin n\pi x}{n\pi} - (2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (2)\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= \left[4\frac{\cos 2n\pi}{n^2\pi^2}\right]$$

$$= \frac{4(-1)^{2n}}{n^2\pi^2}$$

$$a_n = \frac{4}{n^2\pi^2}$$

$$b_n = \int_0^2 x^2 \sin n\pi x dx$$

$$= \left[x^2\left(\frac{-\cos n\pi x}{n\pi}\right) - (2x)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^2$$

$$= \left[\frac{-4\cos 2n\pi}{n\pi} + \frac{2\cos 2n\pi}{n^2\pi^2} - \frac{2\cos 0}{n^2\pi^2}\right]$$

$$b_n = \frac{-4}{n\pi}$$

$$f(x) = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}\cos n\pi x - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi$$

Example 4:-

Find Fourier series for the periodic function

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}, \text{ deduce that } \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Solution:-

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi dx + \int_{0}^{\pi} x dx\right]$$

$$= \frac{1}{\pi}\left[-\pi(x)\,_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi(\pi) + \frac{\pi^2}{2}\right]$$

$$a_0 = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx dx + \int_{0}^{\pi} x \cos nx dx\right]$$

$$a_n = \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\frac{(-1)^n - 1}{n^2}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx dx + \int_{0}^{\pi} x \sin nx dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{\pi(-1)^n}{n}\right] + \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{2\pi(-1)^n}{n}\right]$$

$$b_n = \frac{1-2(-1)^n}{n}$$

$$f(x)= \frac{-\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\frac{(-1)^n-1}{n^2}\cos nx + \frac{1-2(-1)^n}{n}\sin nx\right)$$

$$\begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases} = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\frac{(-1)^n-1}{n^2}\cos nx + \frac{1-2(-1)^n}{n}\sin nx\right)$$

Therefore, the Fourier series at $x_0 = 0$ converges to,

$$\frac{1}{2}[f(x_0-0) + f(x_0+0)]$$

$f(x_0 - 0)=lim_{x\to 0^-} f(x) = -\pi$ &

$f(x_0 + 0) = lim_{x\to 0^+} f(x) = 0$

$\therefore f(0)=\frac{1}{2}(-\pi + 0)= -\frac{\pi}{2}$

Therefore,

Now, take x=0, we get

$$f(0)=-\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{1}{\pi}\frac{(-1)^n-1}{n^2}$$

$$\Rightarrow -\frac{\pi}{2} = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2} .$$

$$\Rightarrow -\frac{\pi^2}{4} = -2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\Rightarrow \frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

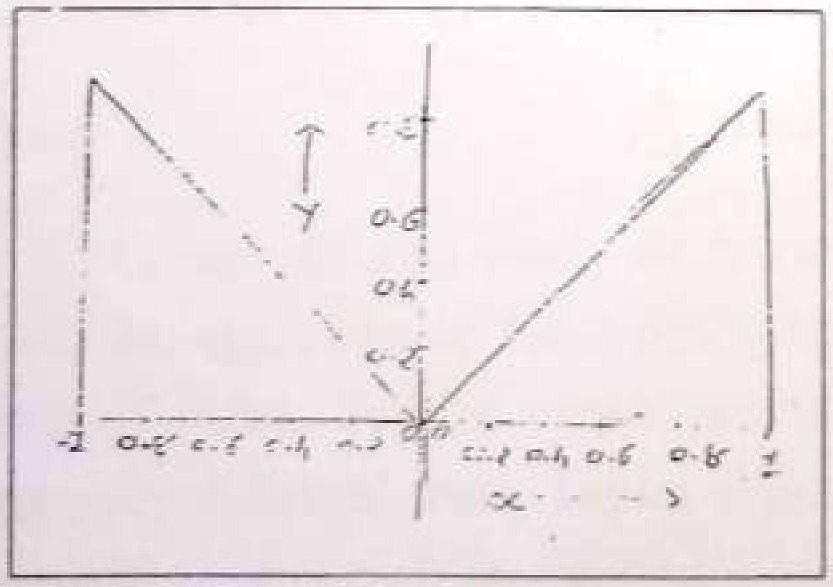
4. Fourier series for Even & Odd Function

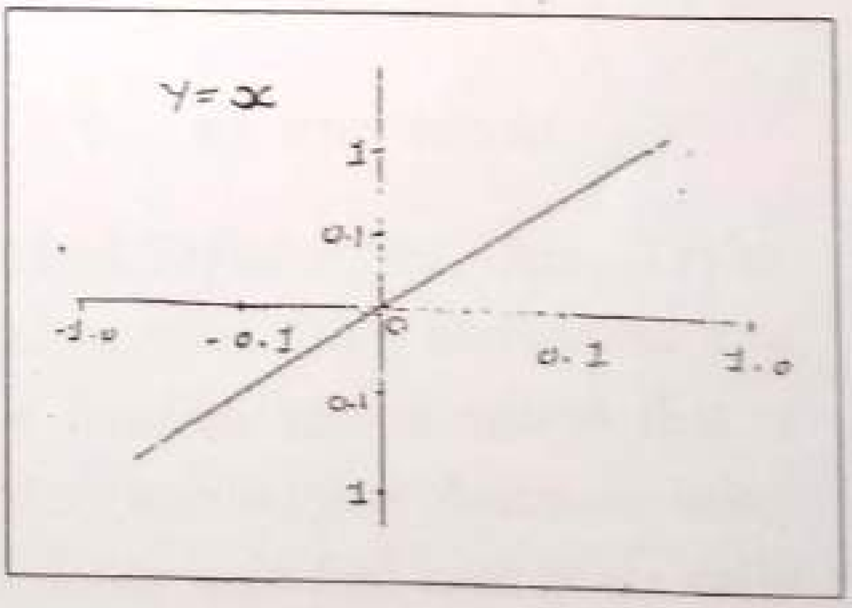
4.1 Definition :-

A function y= f(x) is said to be <u>even</u>, if f (-x) = f(x). The graph of the even function is always symmetrical about the y-axis.

A function y= f(x) is said to be <u>odd</u>, if f (-x) = -f(x). The graph of the odd function is always symmetrical about the origin.

For example, the function f(x) = $|x|$ in [-1 , 1] is even as f(-x)=$|-x|$ = $|x|$=f(x) and the function f(x)=x in [-1,1] is odd as f(-x)= -x = -f(x). The graphs of these functions are shown below:





Note that the graph of f(x) = $|x|$ is symmetrical ab0ut the y-axis and the graph of f(x) = x is symmetrical about the origin

1. If f(x) is even and g(x) is odd, then

- h(x)=f(x) × g(x) is odd
- h(x)=f(x) × g(x) is evèn
- h(x)=g(x) × g(x) is even
  - for example
- h(x)=$x^2 \cos x$ is even , since both x^2 and $\cos x$ are even functions
- .h(x)=$x \sin x$ is even , since x and $\sin x$ are odd functions
- .h(x)=$x^2 \sin x$ is odd,since even and since $x^2$is even and $\sin x$ is odd

2. If f(x) is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

3. If f(x) is odd, then $\int_{-a}^{a} f(x)dx = o$

For example $\int_{-a}^{a} \cos x dx = 2 \int_{0}^{a} \cos x dx$,as $\cos x$ is even

And $\int_{-a}^{a} \sin x dx = 0$, $as \ \sin x \ is \ odd$

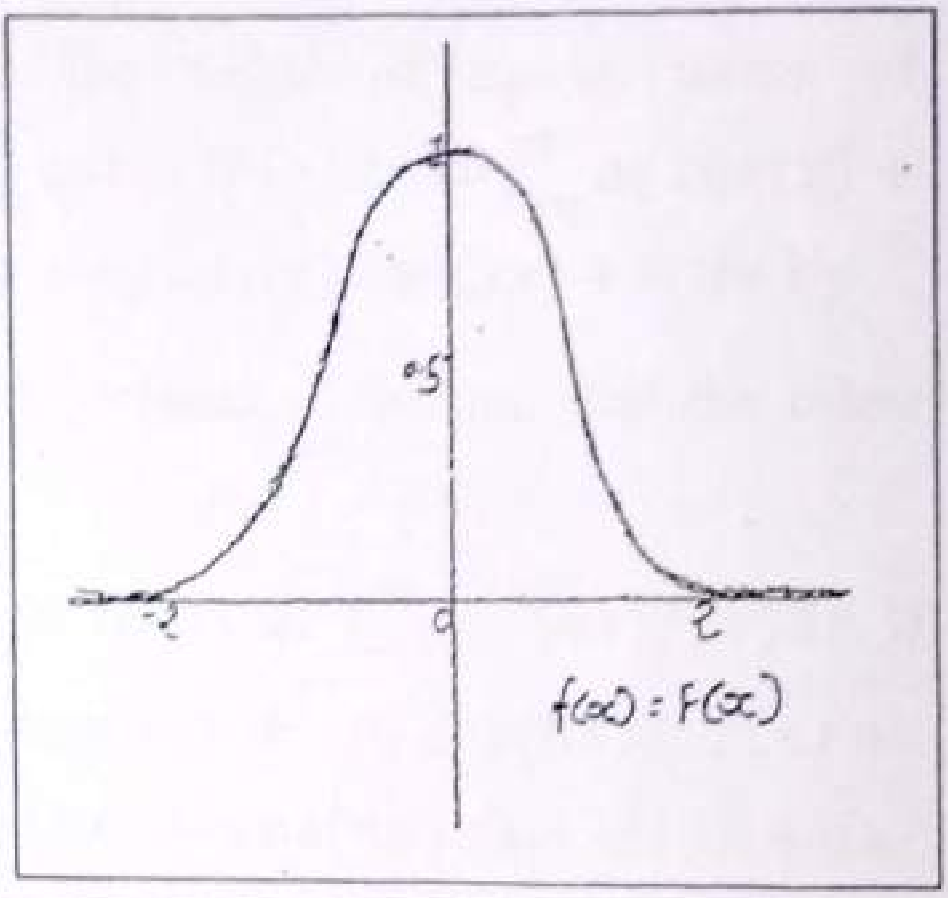
In our calculus class, we have studied Taylor series. Using Taylor series, we approximate functions with polynomials using derivatives at a specified point. Fourier series provide a function approximation that is inherently different from Taylor series; they approximate functions using sines and cosines over an interval. Fourier series were first used in the early 1800s by Joseph Fourier (1768-1830) to describe complicated periodic phenomena. Since a Fourier series uses only sine and cosines, it always creates a periodic function as the approximating function. Consequently, Fourier approximations are often applied to the study of heat flows, oscillations, vibrations, sound and other wave forms that exhibit periodicity. Today, processes associated with Fourier series can be used in speech recognition, music analysis, and in understanding how sound is affected by transmission through cell phones.

A Fourier series is an infinite trigonometric series of the form

Which can be written using summation notation as

$F(x)=a_0 + \sum_{k=1}^{\infty}[a_k \cos(kx) + b_k \sin(kx)]$

Our goal increasing a Fourier series is to approximate a given function with the Fourier series given above by choosing appropriate values for and at right, we see the 2^{nd} order Fourier approximation (blue) to the function $y=e^{-x^2}(red)$.



4.2 Fourier Series for Even Functions

Recal! that if it an even function, an even Fourier series, we will denote it by $F_E(X)$ has only the cosine terms, and can be used to approximate an even function, so $F_E(X) = a_0 + a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(3x) + K$. In this section, we will begin by developing an even Fourier approximation for some general even function f. Later we will expand the process to produce the general Fourier series for arbitrary functions.

Given an arbitrary even function f on the interval $[-\pi, \pi]$, we want to find the function $F_E(X)$ so that f(x)=$F_E(x)$. this means that f(x)=$a_0 + a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(3x) + k$ and, consequentl,

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 + a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(3x) + k \, dx$$

1. Use the equation above to find the value of a_0 in terms of $\int_{-\pi}^{\pi} f(x)dx$

2. Simplify $cos(nx + mx) + cos(nx - mx)$ using the sum and difference identities from trigonometry and use it to evaluate

$$\int_{-\pi}^{\pi} cos(nx)\, cos(mx) dx, when\ m \neq n \text{ and when } m=n.$$

3. Use the result from (2) to find the value of a_1 in terms of $\int_{-\pi}^{\pi} cos(x) f(x) dx$ if $\int_{-\pi}^{\pi} cos(x) f(x) dx = \int_{-\pi}^{\pi} a_0 \, cos(x) + a_1 cos^2(x) + a_2 \, cos(x)\, cos(2x) + a_3 \, cos(x)\, cos(3x) + k \; dx$ by

Multiplying our original function f by cosines, we can find the other coefficients.

4. Generalize to find the value of a_n in terms of $\int_{-\pi}^{\pi} cos(nx) f(x) dx$ if

$$\int_{-\pi}^{\pi} cos(nx) f(x) dx = \int_{-\pi}^{\pi} a_0 \, cos(nx) + a_1 \, cos(nx)\, cos(x) + a_2 \, cos(nx)\, cos(2x) + cos(nx)\, cos(3x) + k \; dx$$

It might help to look at n=2 and n=3 first. This result gives us a rule for finding the coefficients to approximate any even function on the interval $[-\pi, \pi]$.

5. If f(x) $= e^{-x^2}$ on the interval $[-\pi, \pi]$, use an even Fourier series and numerical integration on your calculator to determine the coefficients a_0, a_1, a_2, a_3, a_4 and a_5. Compare the graph of f(x) $= e^{-x^2}$ to that of your series $F_E(x) = a_0 + a_1 \, cos(x) + a_2 \, cos(2x) + k + a_5 \, cos(5x)$ on the interval $[-\pi, \pi]$.

4.3 Fourier Series for Odd Functions

Recall that if it is an odd function. An odd Fourier series has only the sine terms, and can be used to approximate an odd function, so

1. Why is there no b_0 terms in the series $f_0(x)$?

2. Using steps similar to those outlined for even functions develop a rule for finding the coefficients to approximate any odd function on the interval $[-\pi, \pi]$.

3. If on the interval, use an odd Fourier series and numerical integration on your calculator to determine the coefficients.

4.4 General Fourier series

Now we are ready to consider Fourier series for any function. Using steps similar to those used above develop a rule for finding the coefficients to approximate an arbitrary function f on the interval.

In our prior work, we saw how multiplication by $cos(nx)$ and integrating generates the equation

$$\int_{-\pi}^{\pi} cos(nx)f(x)dx = \int_{-\pi}^{\pi} a_0 \cos(nx) + a_1 \cos(nx)\cos(x) + a_2 \cos(nx)\cos(2x) + a_3 \cos(nx)\cos(3x) + k\, dx$$

By evaluating the integrals, we eliminate all but one terms in F_E allowing us to find the value of a_n in terms of the value of $\int_{-\pi}^{\pi} cos(nx)(x)dx$. similarly, we can eliminate all but one term in F_0 by multiplying by $sin(nx)$ and integrating. What we need to consider in the general form is how the sines and cosines interact when we multiply and integrate.

1. What can you say about the value of $\int_{-\pi}^{\pi} cos(nx)\, sin(kx)$ for all $n \neq k$?

2. Apply your technique to determine the coefficients and if $f(x) = \frac{1}{1+e^x}$ on the interval.

3. Compare the graphs of f and F as you increase the number of terms used in the approximation.

5. HALF - RANGE FOURIER SERIES

The Fourier series expansion of the periodic function f(x) of period 2l may contain both sine and cosine terms. Many a time it is required to obtain the Fourier expansion of f(x) in the interval (0, l) which is regarded as half interval. The definition can be extended to the other half in such a manner that the function becomes even or odd. This will result in cosine series or sine series only.

1) Half range cosine series in the interval (0,l).

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right)\right]$$

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

2) Half range cosine series in the interval $(0,\pi)$.

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

3) Half range sine series in the interval (0,l).

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

4) Half range sine series in the interval $(0,\pi)$.

$$F(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

5.1 Sine series:

Suppose f(x) = $\varphi(x)$ is given in the interval (0,l). Then we define

F(x) =$\varphi(-x)$ in (-l, 0).

Hence f(x) becomes an odd function in (-l,l). The Fourier series then is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\rho x}{l})$$

$$\text{Where } b_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\rho x}{l})\,dx \quad \text{.................. (1)}$$

The series (1) is called half – range sine series over (0,l).

Putting l=π in (1), we obtain the half – range sine series of f(x) o over (0,π) given by $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\rho}\int_0^{\rho} f(x)\sin nx dx$$

5.2 Cosine Series:

Let us define

f(x)=

$$\begin{cases} f(x) & in\ (0,l)\ \ ....given \\ f(x) & in\ (-l,0)\ ..... \ in\ order\ to\ make\ the\ function\ even. \end{cases}$$

Then the Fourier series of f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\rho x}{l})$$

$$where,\ a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos(\frac{n\rho x}{l})\,dx \quad \text{...................(2)}$$

The series (2) is called half – range cosine series over (0,l)

Putting l=π in (2), we get

$$F(x)=\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where, $a_0 = \frac{2}{\rho}\int_0^{\rho} f(x)dx$

$$a_n = \frac{2}{\rho}\int_0^{p} f(x)\cos nx\,dx \qquad n=1, 2, 3, \ldots\ldots\ldots$$

5.3 Examples:-

Example1:-

Expand f(x)= x $(\pi - x)$ as half-range sine series over the interval $(0,\pi)$.

Solution: we have,

$$b_n = \frac{2}{\rho}\int_0^{\rho} f(x)\sin nx dx$$

$$= \frac{2}{\rho}\int_0^{\rho} (\rho x - x^2)\sin nx dx$$

Integrating by parts, we get

$$b_n = \frac{2}{\rho}[(\rho x - x^2\left(\frac{-\cos nx}{n}\right) - (\rho - x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^2}\right)]_0^{\rho}$$

$$= \frac{4}{n^3\rho}[1-(-1)^n]$$

The sine series of f(x) is

$$F(x) = \frac{4}{\rho}\sum_{n=1}^{\infty}\frac{1}{n^3}[1-(-1)^n]\sin nx$$

Example:-2

Obtain the cosine series of

$$f(x) = \begin{cases} x, 0 < x < \frac{\rho}{2} \\ \rho - x, \frac{\rho}{2} < x < \rho \end{cases} \qquad \text{over } (0,\rho)$$

Solution: Here

$$a_0 = \frac{2}{\rho}[\int_0^{\rho/2} x dx + \int_{\rho/2}^{\rho} (\rho - x)dx] = \frac{\rho}{2}$$

$a_n = \frac{2}{\rho}[\int_0^{\rho/2} x \cos nxdx + \int_{\rho/2}^{\rho} (\rho - x) \cos nxdx]$

Performing integration by parts and simplifying, we get

$a_n = -\frac{2}{n^2\rho}[1+(-1)^n - 2\cos(\frac{n\rho}{2})] \quad = -\frac{8}{n^2\rho},$

$n = 2,6,10, \ldots..$

Thus, the Fourier cosine series is

$F(x) = \frac{\rho}{4} - \frac{2}{\rho}[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots\ldots..\infty]$

Example3:-

Find half range cosine series to represent f(x)=$\sin x$, 0<x<π, deduce that,

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Solution:

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$= \frac{2}{\pi}[-\cos x]_0^{\pi}$

$= \frac{2}{\pi}[-\cos\pi + \cos 0]$

$=\frac{2}{\pi}[2]$

$a_0 = \frac{4}{\pi}$

$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nxdx$

$=\frac{1}{\pi}\int_0^{\pi} 2\sin x \cos nxdx$

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$$= \frac{1}{\pi}\int_0^{\pi}(sin(x + nx) + sin(x - nx))dx$$

$$= \frac{1}{\pi}\{\int_0^{\pi} sin(1 + n)xdx + \int_0^{\pi} sin(1 - n)xdx\}$$

$$= \frac{1}{\pi}\left\{\left[-\frac{cos(1+n)x}{(1+n)}\right]_0^{\pi} + \left[-\frac{cos(1-n)x}{(1-n)}\right]_0^{\pi}\right\}$$

$$a_n = -\frac{1}{\pi}\left[\frac{(-1)^{1+n}-1}{1+n} - \frac{(-1)^{1+n}-1}{-1+n}\right] = \frac{2}{\pi}\left[\frac{1}{n-1} + \frac{1}{n+1}\right],$$

n is even, n$\neq$ 1

n is odd, $a_n = 0$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} sin\, 2xdx$$

$$= \frac{2}{\pi}\left[\frac{-cos\, 2x}{2}\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\left[\frac{1}{2} - \frac{1}{2}\right] = 0$$

$$sin\, x = \frac{2}{\pi} + \sum_{n=2}^{\infty} -\frac{1}{\pi}\left[\frac{(-1)^{1+n}-1}{1+n} - \frac{(-1)^{1-n}-1}{-1+n}\right] cos\, nx$$

$$f(x)= \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2,4,6,..}^{\infty}\left[\frac{-1}{n-1} + \frac{1}{n+1}\right] cos\, nx$$

$$sin\frac{n\pi}{2} = \frac{2}{\pi} + \frac{2}{\pi}\left[-\left\{\frac{1}{3} - 1\right\} + \left\{\frac{1}{5} - \frac{1}{3}\right\} - \left\{\frac{1}{7} - \frac{1}{5}\right\} + \cdots\right]$$

$$1 = \frac{2}{\pi}\left[1 + 1 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \cdots\right].$$

$$\frac{\pi}{2} = 2 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

5.4 Euler's Formulae

The Fourier series for the function f(x) in the interval $c \le x \le c+2\pi$ is given by

$$F(x) = {}^{a_0}/_2 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where,

$$a_0 = {}^1/_\pi \int_c^{c+2\pi} f(x)dx$$

$$a_n = {}^1/_\pi \int_c^{c+2\pi} f(x) \cos nx dx$$

$$b_n = {}^1/_\pi \int_c^{c+2\pi} f(x) \sin nx dx$$

Where,

a_0, a_n, b_n These values are known as Euler's Formulae.

6. Complex Fourier series

F {f} = $a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

Is called Real Fourier series.

The **Complex Fourier** Series of f is defined to be

F {f}=$\sum_{n=-\infty}^{\infty} c_n e^{inx}$

Where, $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$

Note:-

$e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos x - i \sin x$

$e^{ix} - e^{-ix} = 2i \sin x$

$e^{ix} + e^{-ix} = 2 \cos x$

$\cos x = \frac{e^{ix}+e^{-ix}}{2}$

$\sin x = \frac{e^{ix}-e^{-ix}}{2}$

- Remark

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

$$= \frac{1}{2}[\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx - i\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx]$$

$c_n = \frac{1}{2}(a_n - ib_n)$, n>0

$c_n = \frac{1}{2}(a_{-n} + ib_{-n})$, n<0

$c_n = a_0$, n=0

6.1 Examples:

Example 1:-

Write the complex Fourier transform of

F(x) =$2\sin x - \cos 10x$

Solution:

$$F(x) = 2\frac{e^{ix}-e^{-ix}}{2i} - \frac{e^{10ix}+e^{-10ix}}{2}$$

$$= \frac{1}{i}e^{ix} - \frac{1}{i}e^{-ix} - \frac{1}{2}e^{10ix} - \frac{1}{2}e^{-10ix}$$

$$c_1 = \frac{1}{i},\quad c_{10} = -\frac{1}{2},\quad c_{-1} = -\frac{1}{i},\quad c_{-10} = -\frac{1}{2}$$

Example:-2

Find the real Fourier series of

F(x) = $5\sin x - e^{ix} - ie^{-2ix}$

Solution:

F(x) = $5\sin x - \cos x - i\sin x - i(\cos 2x - i\sin 2x)$

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Type equation here.

Example: - 3

Find the complex Fourier series of

$F(x) = x, -\pi < x < \pi$

Solution:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx$$

$$= \frac{1}{2\pi}\left[-\frac{xe^{-inx}}{in} + \frac{e^{-inx}}{n^2}\Big|_{-\pi}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\left(-\frac{\pi e^{-in\pi}}{in} + \frac{e^{-in\pi}}{n^2}\right) - \left(\frac{\pi e^{in\pi}}{in} + \frac{e^{inx}}{n^2}\right)\right]$$

$$= \frac{1}{2\pi}\left[-\frac{\pi(-1)^n}{in} + \frac{(-1)^n}{n^2} - \frac{\pi(-1)^n}{in} - \frac{(-1)^n}{n^2}\right]$$

$$= -\frac{(-1)^n}{in}, \quad n \neq 0$$

For n=0

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = a_0$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} xdx = 0$$

Therefore complex Fourier series is

$$F\{f\} = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$= \underbrace{0}_{c_0} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{(-1)^{n+1}}{in} e^{inx}$$

Note:-

By a complex trigonometric polynomial, we mean a finite part of

6.2 Parseval's Identity

Parseval's identity for complex Fourier series

$$\sum_{n=-\infty}^{\infty}|c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2 dx$$

Where, $|a+ib|^2 = a^2 + b^2$

Note:-

$|2+3i|^2 = 4+9 = 13$

$|3i|^2 = |2+3i|^2 = 0^2 + 3^2 = 9$

$|i| = \sqrt{0^2+1^2} = 1$

Example 1

$F\{f\} = \sum_{n\neq 0}\frac{(-1)^{n+1}}{in}e^{inx}$

Solution:

Let's apply Parseval's

$$\sum_{n=-\infty}^{\infty}|c_n|^2 = \sum_{n=-\infty,n\neq 0}^{\infty}\left|\frac{(-1)^{n+1}}{in}\right|^2$$

$$\sum_{n=-\infty,n\neq 0}^{\infty}\frac{1}{n^2} = 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}x^2 dx = \frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Example 2

Evaluate:

$$\int_{-\pi}^{\pi} \left|1 - e^{ix} + 3ie^{4ix} - (1+i)e^{6ix} - \cos 4x\right|^2 dx$$

Solution:

Let,

$$F(x) = 1 - e^{ix} + 3ie^{4ix} - (1+i)e^{6ix} - \cos 4x$$

Want

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2$$

$$c_0 = 1, \;\; c_1 = -1, \;\; c_4 = 3i - \frac{1}{2}, \;\; c_{-4} = \frac{1}{2}, \;\; c_6 = -1 - i$$

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi\left(1 + 1 + 9 + \frac{1}{4} + \frac{1}{4} + 2\right)$$

7. Fourier Transform

Let f be defined on $(-\infty, \infty)$

We defined its Fourier transform by

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

Parseval's Identity

$$\int_{-\infty}^{\infty} |\hat{f}(w)|^2 dw = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Fourier Transform:

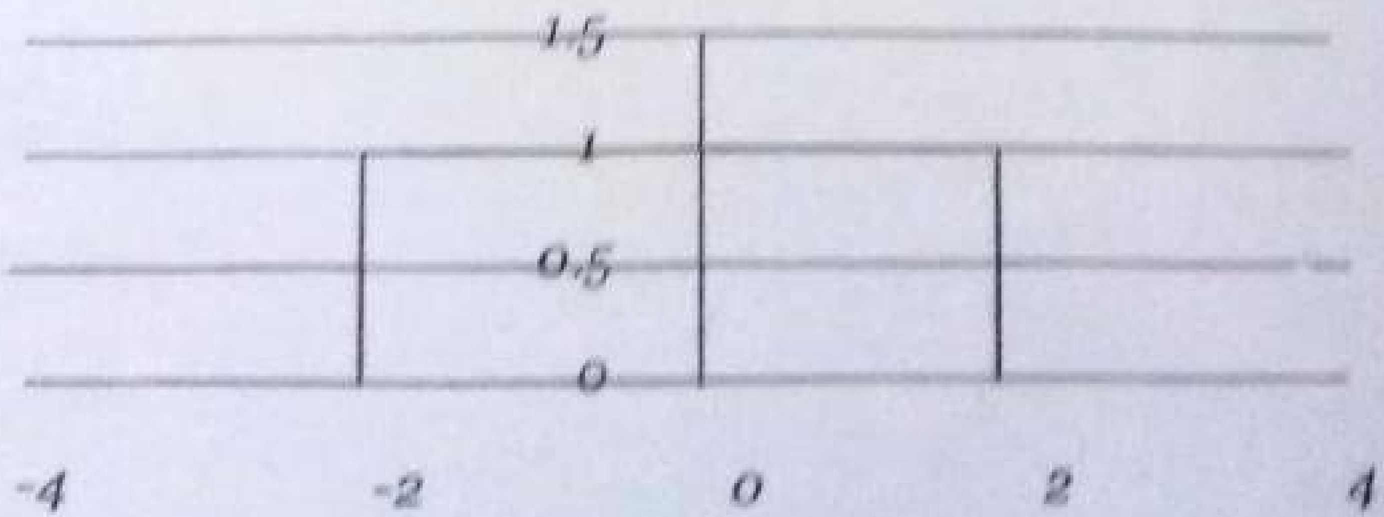
Example 1

Find the Fourier transform of

$$F(x) = \begin{cases} 1: & -2 < x < 2 \\ 0: & otherwise \end{cases}$$

Then apply parseval's identity and see what it gives

Solution:



$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-ixw}}{-iw} \Big|_{-2}^{2} \right)$$

$$= \frac{-1}{iw\sqrt{2\pi}} [e^{-2iw} - e^{2iw}]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\sin 2w}{w} \right)$$

Note here if we are Asked about $\hat{f}(0)$, we take the limit

Let's apply parseval's

$$\int_{-\infty}^{\infty} \frac{2}{\pi} \left(\frac{\sin^2 2w}{w^2} \right) dw = \int_{-2}^{2} dx = 4$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{2}{\pi} \left(\frac{\sin^2 2w}{w^2} \right) dw = 2\pi$$

1) Let's play with

$$\int_{-\infty}^{\infty} \frac{2}{\pi} \left(\frac{\sin^2 2w}{w^2} \right) dw = 2\pi$$

2) Let 2w=t$\Rightarrow$ dw = $\frac{dt}{2}$

$$\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} dt = \pi$$

$$\int_{0}^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$$

3) Let's find

$$\int_{0}^{\infty} \frac{\sin t}{t} dt$$

$$\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}$$

$$\underbrace{-\frac{\sin^2 t}{t}\Big|_0^\infty}_{0} + 2\int_0^\infty \frac{\sin t \cos t}{t}\,dt = \frac{\pi}{2}$$

$$\Rightarrow \int_0^\infty \frac{\sin 2t}{t}\,dt = \frac{\pi}{2}$$

Let 2t=y

$$-\int -\frac{1}{t}$$

$$\int_0^\infty \frac{\sin y}{y}\,dy = 2\pi$$

4) $\int_{-\infty}^{\infty} \frac{\overbrace{\sin y}^{odd}}{\underbrace{y}_{odd}}\,dy = \pi$

Integration by parts

$\sin^2 t \searrow \frac{1}{t^2}$

$2\sin t \cos t \rightarrow \int -\frac{1}{t}$

Note:-

$\hat{f}$ Is continuous regardless of f

$$lim_{w\to\pm\infty} \hat{f}(\infty) = 0$$

$$lim_{n\to\pm\infty} c_n = lim_{x\to\infty} a_n = lim_{x\to\infty} b_n = 0$$

7.1 Fourier Sine and Cosine Transforms

If f is defined on $(0, \infty)$, we define its

Fourier Cosine Transform by

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos wx \, dx$$

And **Fourier Sine Transform**

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin wx \, dx$$

Where did these equations come from?

Recall Fourier transform

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

If f is even

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)[\cos wx - i \sin wx] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos wx \, dx - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overbrace{f(x) \sin wx dx}^{0}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos wx \, dx$$

Note:-

Practically

$\hat{f}_c(w) = \hat{f}(w)$ When f is even.

-i$\hat{f}_s(w) = \hat{f}(w)$ when f is odd.

Note that when f is defined on $(0,\infty)$, we can consider it even or odd.

Example 1

Find $\hat{f}_c(w)$ *and* $\hat{f}_s(w)$ for

$$F(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

Solution:

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 x \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{x \sin wx}{w} + \frac{\cos wx}{w^2} \right) \Big|_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\sin w}{w} + \frac{\cos w}{w^2} - \frac{1}{w^2} \right)$$

Using limits

$$\hat{f}_c(0) = \sqrt{\frac{2}{\pi}} \left(1 - \frac{1}{2} \right) = \frac{1}{2} \sqrt{\frac{2}{\pi}}$$

7.2 Inverse Fourier Transform

Fourier Inverse Transform

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{ixw} dw$$

Fourier Inverse Cosine Transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_c(w) \cos xw \, dw$$

Fourier Inverse Sine Transform

$$F(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_s(w) \sin xw \, dw$$

- **Useful rules**

$$F_c\{f'\} = wF_s\{f\} - \sqrt{\frac{2}{\pi}} f(0)$$

$$F_s\{f'\} = wF_c\{f\}$$

Example 1:-

Find$F_c\{e^{-x}\}$; $F_s\{e^{-x}\}$

Solution:

$F(x) = e^{-x} \Rightarrow f'(x) = -e^{-x}$

Using the rules

$$F_c\{f'\} = wF_s\{f\} - \sqrt{\frac{2}{\pi}}f(0)$$

$$F_c\{-e^{-x}\} = wF_s\{e^{-x}\} - \sqrt{\frac{2}{\pi}}$$

$$= wF_s\{-f'\} - \sqrt{\frac{2}{\pi}}$$

$$= -wF_s\{f'\} - \sqrt{\frac{2}{\pi}}$$

Example 2:-

You are given that

$$F_s\{e^{-x}\} = \sqrt{\frac{2}{\pi}}\frac{w}{w^2+1}$$

$$\Rightarrow e^{-x} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{w}{w^2+1}\sin xw\, dw$$

$$= \frac{2}{\pi}\int_0^\infty \frac{w \sin xw}{w^2+1}dw$$

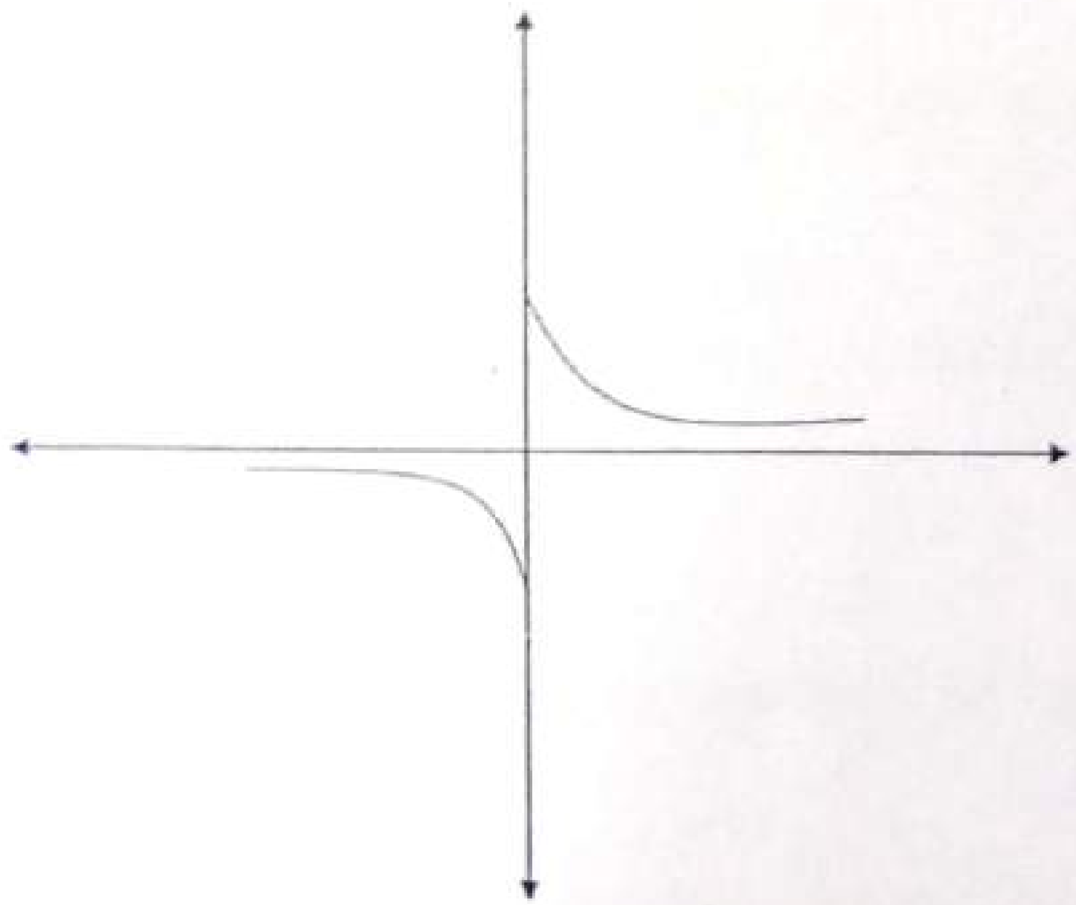
$$x=1 \Rightarrow \int_0^\infty \frac{w \sin xw}{w^2+1}dw = \frac{\pi}{2}e^{-1}$$

The formula of the Fourier Inverse sine Transform

f(x) = $\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(w)\, \sin xw \, dw$ is true when f is continuous at x.

Moreover, recall that $\hat{f}_s(w)$ is computed for odd function f.

If we extend e^{-x} to be odd, we get



Not continuous at x=0 when taking $\hat{f}_s(w)$, so we use dirichlet's Theorem.

8. Parseval's Identity

Consider Fourier series and expand it

$$F(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

Square it

$$f^2(x) = {a_0}^2 + \sum_{n=1}^{N}({a_n}^2 \cos^2 nx + {b_n}^2 \sin^2 nx) + 2\, a_0 \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) + 2a_1 \cos x\, b_1 \sin x + 2a_1 \cos x \sum_{n=2}^{N}(a_n \cos nx + b_n \sin nx) + \cdots + 2a_N \cos Nx b_N \sin Nx$$

Integrate

$$\int_{-\pi}^{\pi} f^2(x)dx = \int_{-\pi}^{\pi}\left\{{a_0}^2 + \sum_{n=1}^{N}({a_n}^2 \cos^2 nx + {b_n}^2 \sin^2 nx) + \ldots\right\} dx$$

$$\Rightarrow \int_{\pi}^{\pi} f^2(x)dx = 2\pi {a_0}^2 + \sum_{n=1}^{N}\left(\pi {a_n}^2 + \pi {b_n}^2\right) + 0$$

Parseval's Identity

Standard form

$$2|a_0|^2 + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$$

General form

$$2|a_0|^2 + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) = \frac{1}{L}\int_{-L}^{L} f^2(x)dx$$

8.1 Examples:-

Example 1:-

F {f} $=\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}sin(2n-1)$ where

$$F(X)=\begin{cases}1 : 0 < x < \pi \\ -1 : -\pi < x < 0\end{cases}$$

L.H.S of Parseval's

$$2(0)+\sum_{n=1}^{\infty}\left(0^2+\left(\frac{4}{(2n-1)\pi}\right)^2\right)=\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

R.H.S of parseval's

$$\frac{1}{\pi}\int_{-\pi}^{\pi}1\,dx=2$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$$

Example 2

$$F\{x^2\}=\frac{1}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2\pi^2}cos(n\pi x),\ x^2$$

Apply parseval's

$$2\left(\frac{1}{3}\right)^2+\sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}=\int_{-1}^{1}x^4dx=\frac{2}{5}$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}=\frac{2}{5}-\frac{2}{9}=\frac{8}{45}$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$$

Example 3

Find $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Now series is given but not f(x).

Solution:

We need f(x) such that

$a_n = \frac{1}{n}$ or $b_n = \frac{1}{n}$

We attempt with f(x) = x since when integrating by parts, we get n^2in the denominator.

Taking f(x) =x

$a_0 = 0$

$a_n = 0$

Integration by parts

x	$cos(n\pi x)$
1	$\frac{-cos(nx)}{n}$
0 →	$+\int \frac{-sin(nx)}{n^2}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$

$\frac{1}{\pi}\left[x \cos nx | _{-\pi}^{\pi}\right] = \frac{-\pi}{n\pi}[(-1)^n + (-1)^n]$

$= \frac{-2(-1)^n}{n}$

Now apply parseva

$$\sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx$$

$$4 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Example 4

Evaluate

$$\int_{-\pi}^{\pi} (2sin^2 5x - cos\, 3x + cos\, 10x)^2 \, dx$$

Solution:

Let f(x) = $2sin^2 3x - cos\, 3x + cos\, 10x$

Want $\int_{-\pi}^{\pi} f^2(x)dx$

According to parseval's

$$\int_{-\pi}^{\pi} f^2(x)dx = \pi\{2{a_0}^2 + \sum_{n=1}^{\infty}({a_n}^2 + {b_n}^2)\}$$

$$= \pi[2(1)^2 + (-1)^2]$$

$$= 3\pi$$

9. Dirichlet's Theorem

If f is a nice function, then

$$F\{f\}(x_0) = \frac{lim_{x \to x_0} f(x) + lim_{x \to x_0} f(x)}{2}$$

Suppose that f is periodic of period 2π and that f is piecewise continuous, that f'_ and f'_+ both exist

Example 1

Suppose

$$F\{x\} = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n} sin\, nx\ ;\ -\pi < x < \pi$$

Plug $x_0 = 0,\ 0 = 0$

Plug $x = \frac{\pi}{2}$

$$\Rightarrow \frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n} sin\left(\frac{n\pi}{2}\right)$$

$$\sum_{n=1, n\ odd}^{\infty} \frac{2(-1)^n}{n} sin\left(\frac{n\pi}{2}\right) = \frac{\pi}{2}$$

$$\sum_{n=1}^{\infty} \frac{2(-1)^{2n}(-1)^{-1}}{2n-1} sin\left(\frac{(2n-1)\pi}{2}\right) = \frac{\pi}{2}$$

$$\sum_{n=1}^{\infty} \frac{-2(-1)^{n-1}}{2n-1} = \frac{\pi}{2}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} = \frac{\pi}{4}$$

Plug $x_0 = \pi$

F {f} (x_0)= 0, since $\frac{lim_{x \to x_0^-} f(x) + lim_{x \to x_0^+} f(x)}{2} = \frac{\pi - \pi}{2} = 0$

10. Application of Simple Fourier series

so far, we've covered basic sine and cosine Fourier series and its special cases with variable explained. In this section, we are going to understand process of Fourier transform using actual example.

10.1Examples of Transformation

Let f(x) be a periodic function:

$$F(x) = \begin{cases} 0. & if \quad -\pi \leq x < 0 \\ 1. & if \quad 0 \leq x < \pi \end{cases}$$

Then find Fourier coefficient and Fourier series.

First of all we need to gather some information before we transform this piecewise function into Fourier series. Since f(x) ranges over $(-\pi, \pi)$ we assume that period T=2π or L=π which means that f(x+2π)=f(x).

Now, we know the period we can compute Fourier coefficient which is a_n and b_n where L=π:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{\pi}{L}nx\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{\pi}{L}nx\right)dx$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

Then,$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0dx + \frac{1}{\pi}\int_{0}^{\pi} 1dx$$

$$= \frac{1}{\pi}(0 + \pi)$$

$$= 1$$

Now, we solve a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{\pi}{\pi}nx\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cos(nx)dx$$

$$= 0 + \frac{1}{\pi}\left(\frac{\sin(nx)}{n}\Big|_{0}^{\pi}\right)$$

$$= \frac{1}{n\pi}(\sin(n\pi) - \sin(0))$$

$$= 0, \quad \text{for n=1, 2, 3 , ...}$$

Now we solve b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{\pi}{\pi}nx\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0dx + \frac{1}{\pi}\int_{0}^{\pi} 1\sin(nx)dx$$

$$= 0 - \frac{1}{\pi}\left(\frac{\cos(nx)}{n}\Big|_{0}^{\pi}\right)$$

$$= -\frac{1}{n\pi}(\cos(n\pi) - \cos(0))$$

Then,

$$b_n = \begin{cases} 0, & if\ n\ is\ even \\ \frac{2}{n\pi}, & if\ n\ is\ odd \end{cases}$$

Therefore, $a_0 = 1, a_n = 0$ and b_n is solved above

Let f(t) be Fourier series, f(t) can be written as:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L}nt\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L}nt\right)$$

$$\sim \frac{1}{2} + \sum_{n=1}^{\infty} 0 + \sum_{n=1}^{\infty} b_n \sin(nt)$$

Since we know that $b_n = 0$ when n is even, then

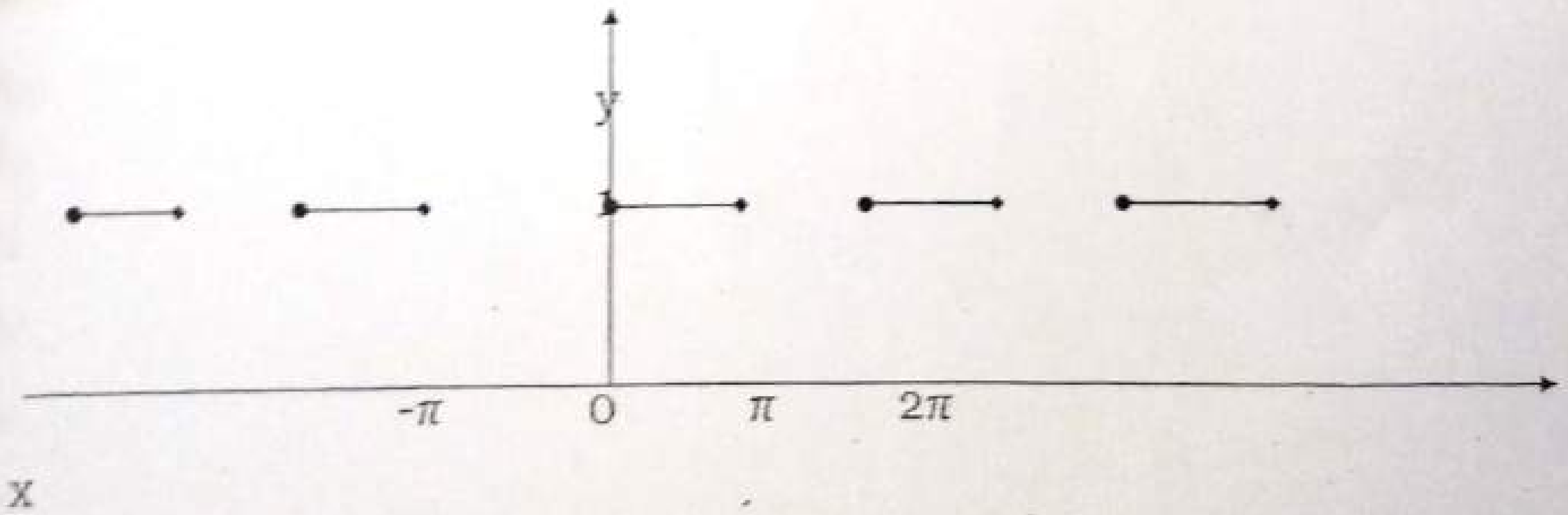
$$f(t) \sim \frac{1}{2} \; if \; n \; is \; even$$

If n is odd then,

$$f(t) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(nt)$$

we know that n is odd, then we can write n as n=2k − 1 where $k \in z$

$$f(t) \sim \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin((2k-1)t)$$



REFERENCES

http:// calvin.enlov.net/blog/2018/01/super –early-preview-Fourier-series

http://en.wikipedia.org/wiki/Fourier _series

http://www.cs.unm.edu/Williams/cs530/symmetry.pdf

http://en.wikiversity.org/wiki/Fourier_series

http://www.inmath.com/fourier -series/helpful -revision.php#prop

APPLICATION OF DERIVATIVE

Shree M P Shah Arts & Science College

Surendranagar

CERTIFICATE

THIS IS TO CERTIFY THAT PROJECT WORK FOR THE SUBJECT OF "APPLICATION OF DERIVATIVE" BY

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1. History:

Newton and Leibniz quite independently of one another, were largely responsible for developing the ideas of integral calculus to the point where hitherto insurmountable problems could be solved by more or less routine methods. The successful accomplishments of these men were primarily due to the fact that they were able to fuse together the integral calculus with the second main branch of calculus, differential calculus.

Isaac Newton
(1642-1727)

Gottfried Leibniz
(1646 -1716)

The central idea of differential calculus is the notation of derivative. Like the integral, the derivative originated from a problem in geometry the problem finding the tangent line at a point of a curve. Unlike the integral. However, the derivative evolved very late in the history of mathematics. The concept was not formulated until early in the 17^{th} century when the French mathematician Pierre de Fermat, attempted to determine the maxima and minima of certain special functions.

Pierre De Fermat
(1601-1665)

2. Definition of Derivative:

We begin with a function f defined at least on some open interval (a, b) on the x-axis. Then we choose a fixed-point x in this interval and introduce the difference quotient

$$\frac{f(x+h)-f(x)}{h}$$

Where the number h, which may be positive or negative (but not zero), is such that x + h also lies in (a, b). The numerator of this quotient measures the change in the function when x changes from x to x + h. The quotient itself is referred to as the average rate of change of f in the interval joining x to x + h.

Now we let h approach zero and see what happens to this quotient. If the quotient approaches some definite value as a limit (which implies that the limit is the same whether h approaches zero through positive values or through negative values), then this limit is called the derivative of f at x and is denoted by the symbol f '(x) (read as "f prime of x"). Thus, the formal definition of f'(x) may be stated as follows:

DEFINITION OF DERIVATIVE: The derivative f '(x) is defined by the equation

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

provided the limit exists. The number f '(x) is also called the rate of change of f at X.

Meaning of derivative: -

- The Derivative is the exact rate at which one quantity changes with respect to another.
- Geometrically, the derivative is the slope of curve at the point on the curve.
- The derivative is often called the "instantaneous" rate of change.
- The derivative of a function represents an infinitely small change the function with respect to one of its variables.
- The Process of finding the derivative is called "differentiation".

3. Application of Derivatives in Various Fields/Science such as in: -

- Biology
- Economics
- Chemistry
- Physics
- Mathematics
- Others (Psychology, sociology & geology)

4. Application of Derivative in Medical and Biology:

Sometimes we may question ourselves why students in biology or medical department still have to take mathematics and even physics. After reading this post, you will understand why.

4.1 Growth Rate of Tumor:

A tumor is an abnormal growth of cells that serves no purpose. There are certain level of a tumor regarding to its malignancy.

The first level is benign tumor. It does not invade nearby tissue or spread to other parts of the body the way cancer can. In most cases, the outlook with benign tumors is very good. But benign tumors can be serious if they press on vital structures such as blood vessels or nerves. Therefore, sometimes they require treatment and other times they do not.

The second level is premalignant or precancerous tumor which is not yet malignant, but is about to become so.

The last level is malignant tumors. These are cancerous tumors, they tend to become progressively worse, and can potentially result in death. Unlike benign tumors, malignant ones grow fast, they are ambitious, they seek out new territory, and they spread (metastasize).

The abnormal cells that form a malignant tumor multiply at a faster rate. Experts say that there is no clear dividing line between cancerous, precancerous and non-cancerous tumors - sometimes determining which is which may be arbitrary, especially if the tumor is in the middle of the spectrum. Some benign tumors eventually become premalignant, and then malignant.

The rate at which a tumor grows is directly proportional to its volume. Larger tumors grow faster and smaller tumors grow slower.

The volume of a tumor is found by using the exponential growth model which is

$$V(t) = V_0 \cdot e^{kt}$$

V_0=initial volume
e=exponential growth

k=growth constant
t=time

In order to find the rate of change in tumor growth, you must take the derivative of the volume equation ($V(t)$)

$$V(t) = V_0 \cdot e^{kt}$$

$$V'(t) = V_0 \cdot e^{kt} \cdot k$$

Because e^{kt} is a complicated function, we use chain rule to derivate it.

$y = e^{kt}$

Let $u = kt$

$y = e^u$

$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt}$

$\frac{dy}{du} = e^u$

$\frac{du}{dt} = k$

$\frac{dy}{dt} = k\,e^u$

$$\frac{dy}{dt} = k\,e^{kt}$$

From the calculation above, we know that the derivative of e^{kt} is $k \cdot e^{kt}$

$$V'(t) = V_0 \cdot k \cdot e^{kt}$$

Because $V(t)$ itself is equal to $V_0 \cdot e^{kt}$ we may conclude

$$V'(t) = k \cdot V$$

- There is the example to prove this theory:

4.2 Larger tumor:

Find the rate of change of a tumor when its initial volume is 10 cm³ with a growth constant of 0.075 over a time period of 7 years

$$V(t) = V_0 \cdot e^{kt}$$

$$V(7) = 10 \times 2.178^{(0.075)7}$$

$$V(7) = 15.05cm^3$$

$$V'(t) = k \cdot V$$

$$V'(t) = 0.075 \times 15.05$$

$$V'(t) = 1.13\, cm^3/year$$

Then let's calculate the rate of change of smaller tumor with the same growth constant and time period.

4.3 Smaller tumor:

Find the rate of change of a tumor when its initial volume is 2 cm^3 with a growth constant of 0.075 over a time period of 7 years

$$V(t) = V_0 \cdot e^{kt}$$

$$V(7) = 2 \times 2.178^{(0.075)7}$$

$$V(7) = 3.01cm^3$$

$$V'(t) = k \cdot V$$

$$V'(t) = 0.075 \times 3.01$$

$$V'(t) = 0.23cm^3/year$$

With this calculation we know how important it is to detect a tumor as soon as possible. It is crucial to give a right treatment that will stop or slow down the growth of the tumor because bigger tumor intend to grow faster and, in some case, becoming a cancer that have a small chance to cured.

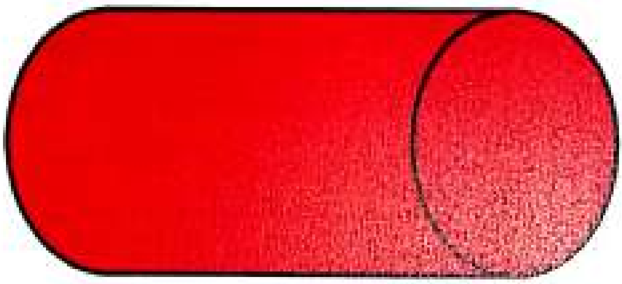
4.4 Blood Flow:

High blood pressure can affect the ability of the arteries to open and close. If your blood pressure is too high, the muscles in the artery wall will respond by pushing back harder. This will make them grow bigger, which makes your artery walls thicker. Thicker arteries mean that there is less space for the blood to flow through. This will raise your blood pressure even further.

Due to fat and cholesterol plaque that cling to the vessel, it becomes constricted. If an artery bursts or becomes blocked, the part of the body that gets its blood from that artery will be starved of the energy and oxygen it needs and the cells in the affected area will die.

If the burst artery supplies a part of the brain then the result is a stroke. If the burst artery supplies a part of the heart, then that area of heart muscle will die, causing a heart attack.

We can calculate the velocity of the blood flow and detect if there are something wrong with the blood pressure or the blood vessel wall.
In this case, we portrait the blood vessel as a cylindrical tube
with radius R and length L as illustrated below



Because of the friction at the walls of the vessel, the velocity of the blood is not the same in every point. The velocity of the blood in the center of the vessel is faster than the flow of the blood near the wall of the vessel. The velocity is decreases as the distance of radius from the axis (center of the vessel) increases until v become 0 at the wall.

The relationship between velocity and radius is given by the law of laminar flow discovered by the France Physician Jean-Louis- Marie Poiseuille in 1840. This state that

$$V = \frac{P}{4\eta L}(R^2 - r^2)$$

V = initial volume
η= viscosity of the blood
P =Pressure difference between the ends of the blood vessel
L = length of the blood vessel
R = radius of the blood vessel
r = radius of the specific point inside the blood vessel that we want to know.

To calculate the velocity gradient or the rate of change of the specific point in the blood vessel we derivate the law of laminar flaw

$$V = \frac{P}{4\eta L}(R^2 - r^2)$$

$$V' = \frac{d}{dr}[\frac{P}{4\eta L}(R^2 - r^2)] = \frac{P}{4\eta L} \cdot \frac{d}{dt}(R^2 - r^2)$$

$$V' = \frac{P}{4\eta L}(0 - 2r)$$

$$V' = -\frac{2rP}{4\eta L}$$

Example:-

The left radial artery radius is approximately 2.2 mm and the viscosity of the blood is 0.0027 Ns/m². The length of this vessel is 20 mm and pressure differences are 0.05 N. What is the velocity gradient at r = 1 mm from center of the vessel?

$$V' = -\frac{2rP}{4\eta L}$$

$$V' = \frac{-2.1 \times 10^{-3} \times 0.05}{4 \times 0.0027 \times 20 \times 10^{-3}}$$

$$V' = \frac{-10^{-4}}{2.16 \times 10^{-4}}$$

$$V' = -0.46\ m/s$$

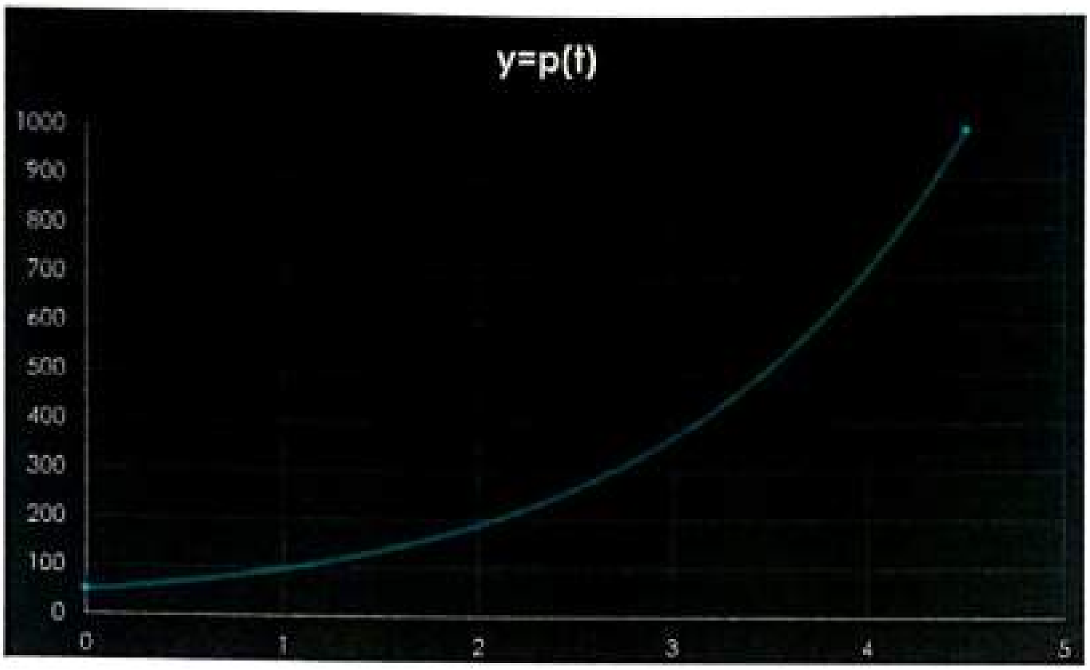
So, we can conclude that the velocity gradient is -0.46 m/s. if the gradient of velocity is too high then the person may have a constriction in his/her blood vessel and needs further examination and treatment.

4.5 Population models

The population of a colony of plants, or animals, or bacteria, or humans, is often described by an equation involving a rate of change (this is called a "differential equation"). For instance, if there is plenty of food and there are no predators, the population will grow in proportion to how many are already there:

$$\frac{dp}{dt} = rp$$

Where r is constant. It's not hard to check that the function $p(t) = p_0 e^{rt}$



$$p_0 = 50, \quad r = 0.65$$

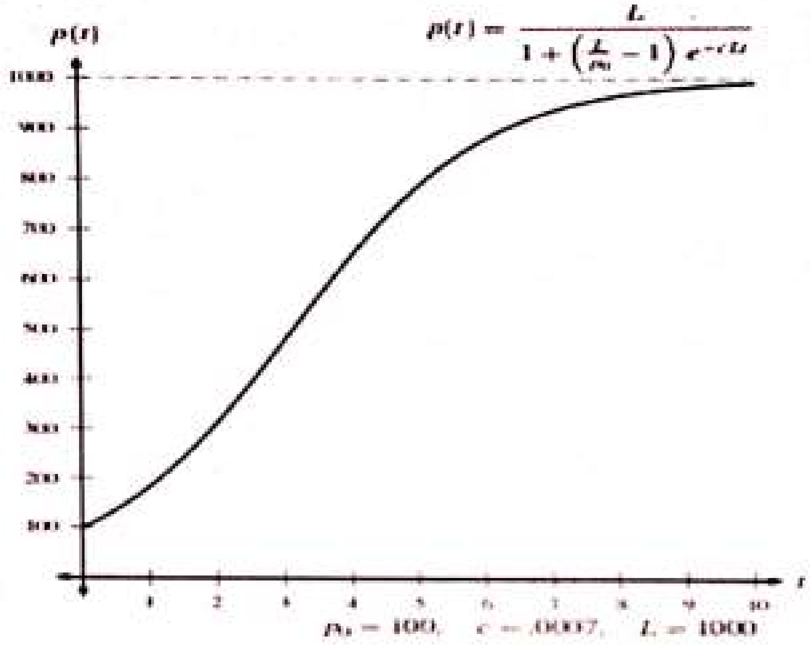
Satisfies this differential equation, where p_0 is the starting population. Colonies tend to grow exponential until they run out of space food or run into predators.

When there are limits on the food supply, the population is often governed by the logistic

EQUATION: -

$$\frac{dp}{dt} = cp(L - p)$$

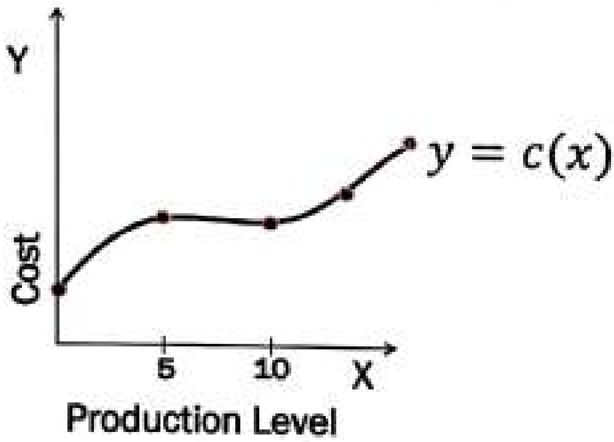
Where c and L are constant. The population grows exponentially for a while, and then levels off at a horizontal asymptote of L



The logistic equation also governs the growth of epidemics, as well as for the example, the frequency of certain genes in a population.

5. Application of Derivative to Business and Economics:

In recent years, economic decision making has become more and more mathematically oriented. Faced with huge masses of statistical data, depending on hundreds or even thousands of different variables, business analysts and economists have increasingly turned to mathematical methods to help them describe what is happening, predict the effects of various policy alternatives, and choose reasonable courses of action from the myriad of possibilities. Among the mathematical methods employed is calculus. In this section we illustrate just a few of the many applications of calculus to business and economics. All our applications will center on what economists call the theory of the firm. In other words, we study the activity of a business (or possibly a whole industry) and restrict our analysis to a time period during which background conditions (such as supplies of raw materials, wage rates, and taxes) are fairly constant. We then show how derivatives can help the management of such a firm make vital production decisions.



Management, whether or not it knows calculus, utilizes many functions of the sort we have been considering. Examples of such functions are
$C(x)$ = cost of producing x units of the product,
$R(x)$ = revenue generated by selling x units of the product,
$P(x) = R(x) - C(x)$ = the profit (or loss) generated by producing and (selling x units of the product.)

Note that the functions $C(x)$, $R(x)$, and $P(x)$ are often defined only for non-negative integers, that is, for $x = 0, 1, 2, 3,$ The reason is that it does not make sense to speak about the cost of producing −1 car or the revenue generated by selling 3.62 refrigerators. Thus, each function may give rise to a set of discrete points on a graph, as in Figure. In studying these functions, however, economists usually draw a smooth curve through the points and assume that C(x) is actually defined for all positive x. Of course, we must often interpret answers to problems in light of the fact that x is, in most cases, a nonnegative integer.

Cost Functions: If we assume that a cost function, C(x), has a smooth graph as in Figure, we can use the tools of calculus to study it. A typical cost function is analyzed in Example 1.

5.1 Marginal Cost Analysis:

EXAMPLE:1:-

Suppose that the cost function for a manufacturer is given by
$C(X) = (10^{-6})X^3 - 0.003X^2 + 5X + 1000$ dollars.

a) Describe the behavior of the marginal cost.

b) Sketch the graph of C(x).

SOLUTION:-

The first two derivatives of C(x) are given by

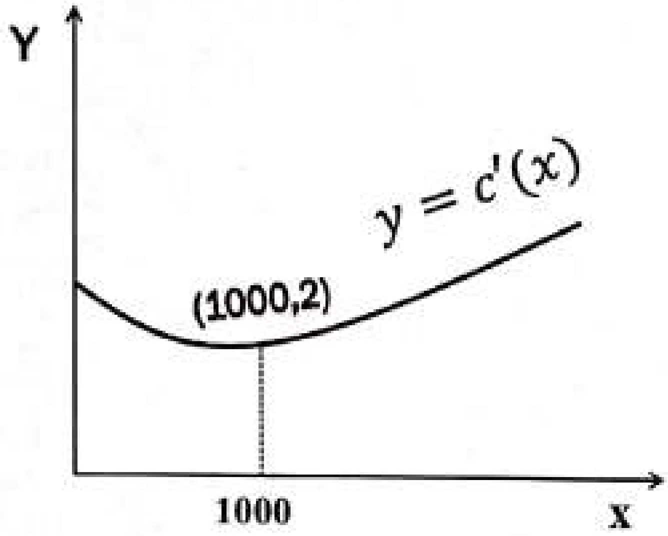
$$C'(X) = (3 \times 10^{-6})X^2 - 0.006X + 5$$
$$C''(X) = (6 \times 10^{-6})X - 0.006$$

Let us sketch the marginal cost C'(x) first. From the behavior of C'(x), we will be able to graph C(x). The marginal cost function$y = (3 \times 10^{-6})X^2 - 0.006X + 5$ has as its graph a parabola that opens upward. Since $y = C''(X) = 0.000006(X - 1000)$, we see that the parabola has a horizontal tangent at $X = 1000$ So, the minimum value of C'(x) occurs at $X = 1000$. The corresponding y-coordinate is

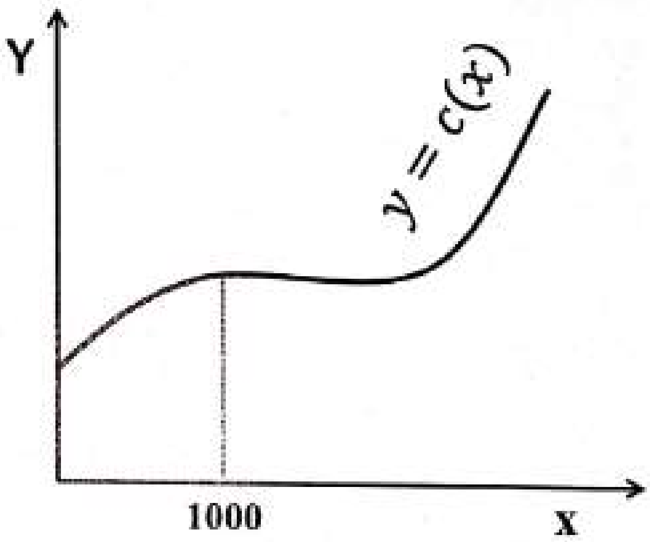
$$(3 \times 10^{-6})(1000)^2 - 0.006 \times (1000) + 5 = 3 - 6 + 5 = 2$$

The graph of $y = C'(x)$ is shown in Figure. Consequently, at first, the marginal cost decreases. It reaches a minimum of 2 at production level 1000 and increases thereafter. This answers part (a). Let us now graph C(x). Since the graph shown in Figure is the graph of the derivative of C(x), we see that C'(x) is never zero, so there are no relative extreme points. Since C'(x) is always positive, C(x) is always increasing (as any cost curve should).

Moreover, since C'(x) decreases for x less than 1000 and increases for x greater than 1000, we see that C(x) is concave down for x less than 1000, is concave up for x greater than 1000, and has an inflection point at x = 1000. The graph of C(x) is drawn in Figure. Note that the inflection point of C(x) occurs at the value of x for which marginal cost is a minimum.



A marginal cost functions.



A cost functions.

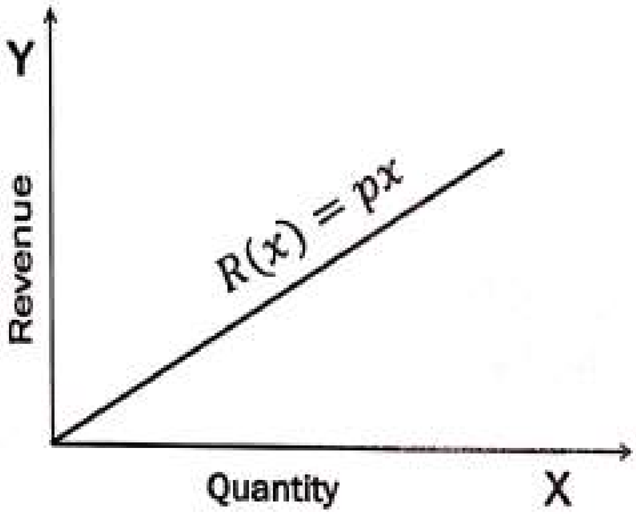
Actually, most marginal cost functions have the same general shape as the marginal cost curve of Example 1. For when x is small, production of additional units is subject to economies of production, which lowers unit costs. Thus, for x small, marginal cost decreases. However, increased production eventually leads to overtime, use of less efficient, older plants, and competition for scarce raw materials. As a result, the cost of additional units will increase for very large x. So, we see that C'(x) initially decreases and then increases.

Revenue Functions In general, a business is concerned not only with its costs, but also with its revenues. Recall that, if R(x) is the revenue received from the sale of x units of some commodity, then the derivative R'(x) is called the marginal revenue. Economists use this to measure the rate of increase in revenue per unit increase in sales.

If x units of a product are sold at a price p per unit, the total revenue R(x) is given by

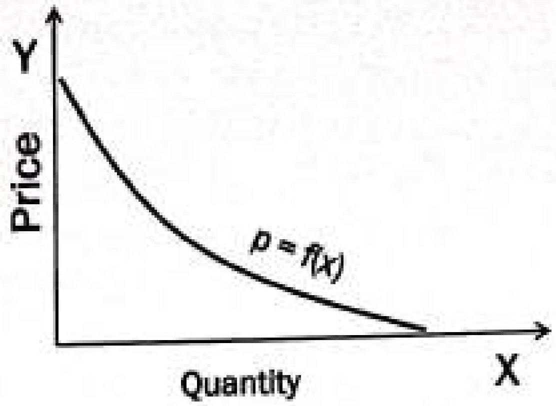
$$R(x) = x \cdot p$$

If a firm is small and is in competition with many other companies, its sales have little effect on the market price. Then, since the price is constant as far as the one firm is concerned, the marginal revenue R'(x) equals the price p [that is, R'(x) is the amount that the firm receives from the sale of one additional unit]. In this case, the revenue function will have a graph Revenue as in Figure.



A revenue curves.

An interesting problem arises when a single firm is the only supplier of a certain product or service, that is, when the firm has a monopoly. Consumers will buy large amounts of the commodity if the price per unit is low and less if the price is raised. For each quantity x, let f(x) be the highest price per unit that can be set to sell all x units to customers. Since selling greater quantities requires a lowering of the price, f(x) will be a decreasing function. Figure shows a typical demand curve that relates the quantity demanded, x, to the price, $p = f(x)$.



A demand curves.

The demand equation $p = f(x)$ determines the total revenue function. If the firm wants to sell x units, the highest price it can set is f(x) dollars per unit, and so the total revenue from the sale of x units is

$$R(x) = xp = xf(x) \quad \text{..........} \ (1)$$

The concept of a demand curve applies to an entire industry (with many producers) as well as to a single monopolistic firm. In this case, many producers offer the same product for sale. If x denotes the total output of the industry, f(x) is the market price per unit of output and x f(x) is the total revenue earned from the sale of the x units.

5.2 Maximizing Revenue:

EXAMPLE: 2 The demand equation for a certain product is $p = 6 - \frac{x}{2}$ dollars Find the level of production that results in maximum revenue.

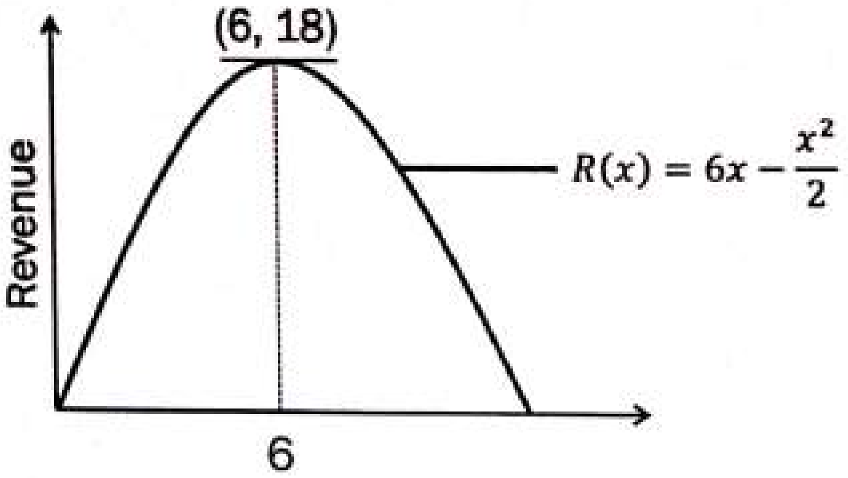
SOLUTION:

In this case, the revenue function R(x) is

$$R(x) = xp = x\left(6 - \frac{x}{2}\right)$$
$$= 6x - \frac{x^2}{2} \text{ dollars.}$$

The marginal revenue is given by

$$R'(x) = 6 - x$$



Maximizing revenue.

The graph of R(x) is a parabola that opens downward. (see figure) It has a horizontal tangent precisely at those x for which R′(x) = 0 that is, for those x at which marginal revenue is 0. The only such x is x = 6. The corresponding value of revenue is

$$R(x) = 6 \cdot 6 - \frac{(6)^2}{2} = 18 \text{ dollars}$$

Thus, the rate of production resulting in maximum revenue is $x = 6$, which results in total revenue of 18 dollars.

Profit Functions: Once we know the cost function C(x) and the revenue function R(x), we can compute the profit function P(x) from

$$P(x) = R(x) - C(x)$$

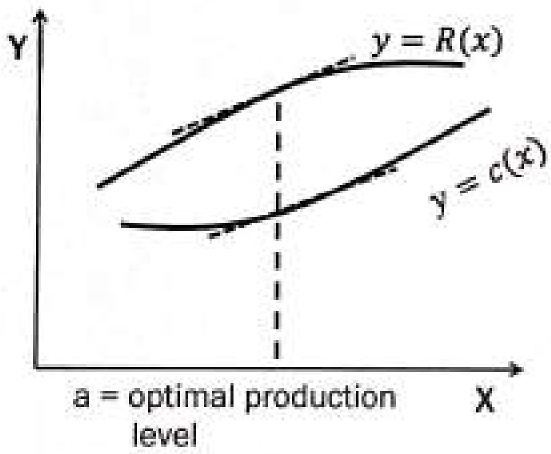
Setting Production Levels: Suppose that a firm has cost function C(x) and revenue function R(x). In a free-enterprise economy the firm will set production x in such a way as to maximize the profit function

$$P(x) = R(x) - c(x)$$

We have seen that if P(x) has a maximum at $x = a$, then $P'(a) = 0$In other words, since

$$P'(X) = R'(X) - C'(X)$$
$$R'(a) = C'(a) = 0$$
$$R'(a) = C'(a)$$

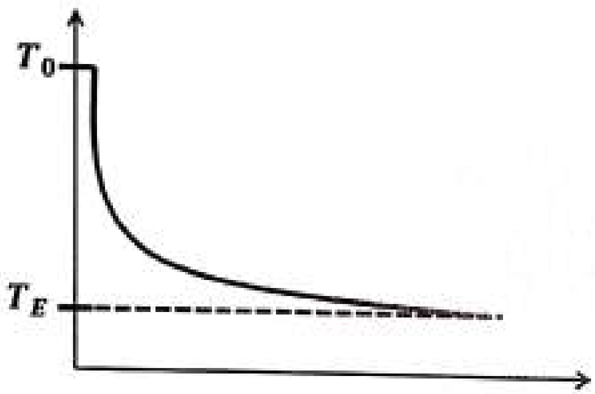
Thus, profit is maximized at a production level for which marginal revenue equals marginal cost. (See Figure)



6. Application of Derivative in Chemistry:

The change in temperature

- An object's temperature over time will approach the temperature of its surroundings (the medium).
- The greater the difference between the object's temperature and the medium's temperature, the greater the rate of change of the object's temperature.
- This change is a form of exponential decay.



6.1 Newton's Law of Cooling.

- It is a direct application for differential equations.
- Formulated by Sir Isaac Newton.
- Has many applications in our everyday life.
- Sir Isaac Newton found this equation behaves like what is called in Math (differential equations) so his used some techniques to find its general solution.

6.2 Derivation of Newton's Law of Cooling:

- Newton's observations:

He observed that observed that the temperature of the body is proportional to the difference between its own temperature and the temperature of the objects in contact with it.

- Formulating:

First order separable DE

- Applying differential calculus:

$$\frac{dT}{dt} = -k(T - T_E)$$

Where k is the positive proportionality constant

- By separation of variables we get

$$\frac{dT}{(T - T_E)} = -kdt$$

- By integrating both sides we get

$$\ln(T - T_E) + C = -kt$$

- At time (t=0) the temperature is T_0

$$C = -\ln(T_0 - T_E)$$

- By substituting $C = -\ln(T_0 - T_E)$ we get

$$\ln\frac{(T - T_E)}{(T_0 - T_E)} = -kt$$

$$\boldsymbol{T = T_E + (T_0 - T_E)e^{-kt}}$$

6.3 Applications on Newton's Law of Cooling:

Investigations.

≫It can be used to determine the time of death.

≫Solar water Heater.

Computer manufacturing.

≫Processors.
≫Cooling systems.

≫Calculating the Surface area Of an object.

6.4 Applications of Newton's Law of Cooling in Investigations in A Crime Scene:

The police came to a house at 10:23 am were a murder had taken place. The detective measured the temperature of the victim's body and found that it was 26.7°C. Then he used a thermostat to measure the temperature of the room that was found to be 20°C through the last three days. After an hour he measured the temperature of the body again and found that the temperature was 25.8°C. Assuming that the body temperature was normal (37°C), what is the time of death?

Solution:

$$T = T_E + (T_0 - T_E)e^{-kt}$$

Let the time at which the death took place be x hours before the arrival of the police men.

Substitute by the given values

$$T(x) = 26.7 = 20 + (37 - 20)e^{-kx}$$
$$T(x+1) = 25.8 = 20 + (37 - 20)e^{-k(x+1)}$$

Solve the 2 equations simultaneously

$$0.394 = e^{-kx}$$
$$0.341 = e^{-k(x+1)}$$

By taking the logarithmic function

$$\ln(0.394) = -kx \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$\ln(0.341) = -k(x+1) \ldots\ldots\ldots\ldots\ldots (2)$$

By dividing (1) by (2)

$$\frac{\ln(0.394)}{\ln(0.341)} = \frac{-kx}{-k(x+1)}$$

$$0.8657 = \frac{x}{x+1}$$

Thus, x≅7 hours

Therefore, the murder took place 7 hours before the arrival of the detective which is at 3:23 pm

6.5 Applications of Newton's Law of Cooling in Processor Manufacturing:

A global company such as Intel is willing to produce a new cooling system for their processors that can cool the processors from a temperature of 50°C to 27°C in just half an hour when the temperature outside is 20°C but they don't know what kind of materials they should use or what the surface area and the geometry of the shape are. So, what should they do?

Simply they have to use the general formula of Newton's law of cooling

$$\boldsymbol{T = T_E + (T_0 - T_E)e^{-kt}}$$

And by substituting the numbers they get

$$27 = 20 + (50 - 20)e^{-0.5k}$$

Solving for k, we get

$$K = 2.9$$

so, they need a material with K=2.9 (k is a constant that is related to the heat capacity, thermodynamics of the material and also the shape and the geometry of the material)

7. Application of Derivative in Physics:

Derivatives with respect to time:

In physics, we are often looking at how things change over time:

1.**Velocity** is the derivative of position with respect to time:

$$v(t) = \frac{d}{dt}(x(t))$$

2.**Acceleration** is the derivative of velocity with respect to time:

$$a(t) = \frac{d}{dt}\big(v(t)\big) = \frac{d^2}{dt^2}\big(x(t)\big)$$

3.**Momentum** (usually denoted p) is mass times velocity, and **force** (F) is mass times acceleration, so the derivative of momentum is

$$\frac{dp}{dt} = \frac{d}{dt}(mv) = m\frac{dv}{dt} = ma = F$$

One of Newton's laws says that for every action there is an equal and opposite reaction, meaning that if particle 2 puts force F on particle 1, then particle 1 must put force −F on particle 2. But this means that the (momentum is constant), since

$$\frac{d}{dt}(p_1 + p_2) = \frac{dp_1}{dt} + \frac{dp_2}{dt} = F - F = 0$$

This is the **law of conservation of momentum.**

Derivatives with Respect to Position:

In physics, we also take derivatives with respect to x.

1. For so called "conservative" forces, there is a function V(x) such that the force depends only on position and is minus the derivative of V, namely $F(x) = -\frac{dV(x)}{dx}$ The function $v(x)$ is called the **potential energy.** For instance, for a mass on a spring the potential energy is $\frac{1}{2}kx^2$, where k is a constant and the force is $-k\, x$.
2. The **kinetic energy** is $\frac{1}{2}mv^2$. Using the chain rule, we find that the **total energy** is

$$\frac{d}{dt}\left(\frac{1}{2}mv^2 + V(x)\right) = mv\frac{dv}{dt} + V'(x)\frac{dx}{dt} = mva - Fv = (ma - F)v = 0$$

since F=ma. This means that the total energy never changes.

These are just a few of the examples of how derivatives come up in physics. In fact, most of physics, and especially electromagnetism and quantum mechanics, is governed by differential equations in several variables.

7.1 Elasticity of Demand

The elasticity of demand E, is the percentage rate of decrease of demand per percentage increase in price. We obtain it from the demand equation according to the following formula:

$$E = \frac{dq}{dp} \cdot \frac{p}{q}$$

Where the demand equation expresses demand q, as a function of unit price p, we say that demand has unit elasticity if E=1.

To find the unit price that maximizes revenue, we express E as a function of p, set E=1, and then solve for p.

Example: -

Suppose that the demand equation $q = 20{,}000 - 2p$.

Then $$E = -(-2)\frac{p}{20{,}000-2p} = \frac{p}{10{,}000-p}$$

If $p = 2000$, then $E = \frac{1}{4}$, and demand is inelasticity at this price.
If $p = 8000$, then $E = 4$, and demand is elasticity at this price.
If $p = 5000$, then $E = 1$, and the demand has unit elasticity at this price.

8. Application of Derivative in Mathematics:

Applications of Maxima and Minima: Optimization Problems:

We solve **optimization problems** of the following form: Find the values of the unknowns x, y, . . . maximizing (or minimizing) the value of the **objective function** f, subject to certain **constraints.** The constraints are equations and inequalities relating or restricting the variables x, y, ...

To solve such a problem, we use the constraint equations to write all of the variables in terms of one chosen variable, substitute these into the objective function f, and then find extrema as above. (We use any constraint inequalities to determine the domain of the resulting function of one variable.) Specifically:

1. **Identify the unknown(s):**
   These are usually the quantities asked for in the problem.
2. **Identify the objective function.**
   This is the quantity you are asked to maximize or minimize.
3. **Identify the constraint(s).**
   These can be equations relating variables or inequalities expressing limitations on the values of variables.
4. **State the optimization problem.**
   This will have the form "Maximize [minimize] the objective function subject to the constraint(s)."
5. **Eliminate extra variables.**
   If the objective function depends on several variables, solve the constraint equations to express all variables in terms of one particular variable. Substitute these expressions into the objective function to rewrite it as a function of a single variable. Substitute the expressions into any inequality constraints to help determine the domain of the objective function.
6. **Find the absolute maximum (or minimum) of the objective function.**

Example:

Here is a maximization problem:

Maximize $A = xy$ Objective Function

subject to $x + 2y = 100$,
$x \geq 0$, and
$y \geq 0$ Constraints

Let us carry out the procedure for solving. Since we already have the problem stated as an optimization problem, we can start at Step 5.

5.Eliminate extra variables.

We can do this by solving the constraint equation $x + 2y = 100$ for x (getting $x = 100 - 2y$) and substituting in the objective function and the inequality involving x:

6.Find the Absolut maximum (or minimum) of the objective function: -

Now, we have to find the maximum value of $A = 100y - 2y^2$.

Taking derivative of A with respect to y,

$$\frac{dA}{dy} = \frac{d}{dy}(100y - 2y^2) = 100 - 4y$$

For extreme points,

$$\frac{dA}{dy} = 0 \qquad 100 - 4y = 0$$

$$y = \frac{100}{4} \qquad y = 25$$

Put value of y in constant x, $x + 2y = 100$.

$x = 100 - 2y$
$x = 100 - 2(25)$
$x = 100 - 50$
$x = 50$

Thus, extreme point is (50,25).

Maximum value of objective function,

$A = xy$
$A = (50)(25)$
$A = 1250$

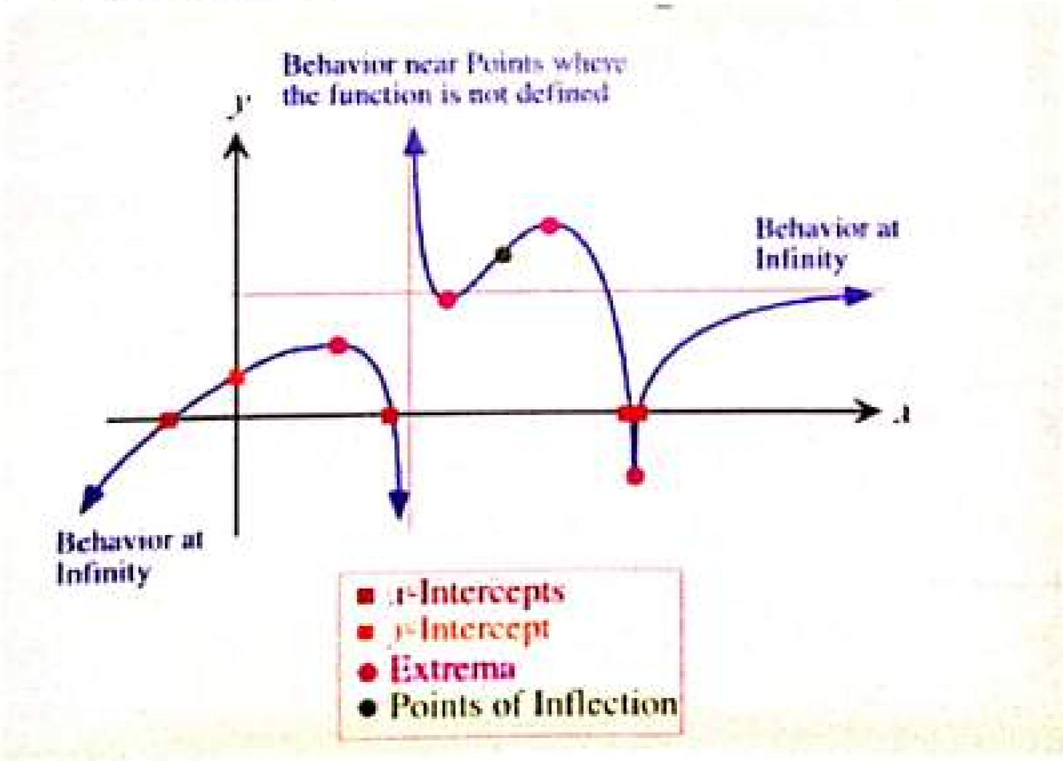
Maximum, $A = 1250$

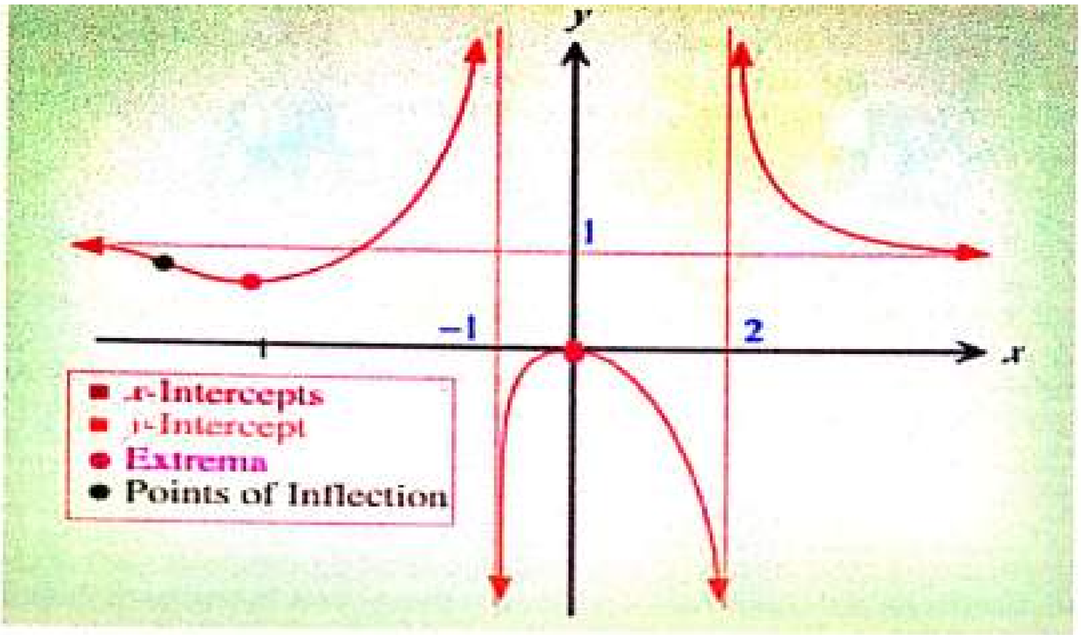
8.1 Analyzing Graphs:

We can use graphing technology to draw a graph, but we need to use differential calculus to understand what we are seeing. The most interesting features of a graph are the following.

Features of a Graph

1. **The x- and y-intercepts:** If $y = f(x)$, find the x-intercept(s) by setting $y = 0$ and solving for x; find the y-intercept by setting $x = 0$.
2. **Relative extrema:** Use the processer to find relative extrema and locate the relative extrema.
3. **Points of inflection:** Set $f''(x) = 0$ and solve for x to find candidate points of inflection.
4. **Behavior near points where the function is not defined:** If f(x) is not defined at x, consider $\lim_{x\to -} f(x)$and $\lim_{x\to +} f(x)$to see how the graph of f approaches this point.
5. **Behavior at infinity:** Consider $\lim_{x\to -\infty} f(x)$ and $\lim_{x\to +\infty} f(x)$if appropriate, to see how the graph of f behaves far to the left and right.





To analyze this, we follow the procedure at left:

1. **The x and y-intercepts:** Setting y = 0 and solving for x gives x = 0. This is the only x-intercept. Setting x = 0 and solving for y gives y = 0: the y-intercept.
2. **Relative extrema:** The only extrema are stationary points found by setting f '(x) = 0 and solving for x, giving x = 0 and x = - 4. The corresponding points on the graph are the relative maximum (0, 0) and the relative minimum at (- 4, 8/9).
3. **Points of inflection:** Solving f "(x) = 0 analytically is difficult, so we can solve it numerically (plot the second derivative and estimate where is crosses the x-axis) and find that the point of inflection lies at x ≈ - 6.1072.
4. **Behavior near points where the function is not defined:** The function is not defined at x = - 1 and x = 2. The limits as x approaches these values from the left and right can be inferred from the graph:

Other Application of Derivatives in Mathematics:

- Approximation by differentials and newton's method
- Monotonic functions, relative and absolute extrema of functions
- Convex functions, inflection points and asymptotes
- Curve sketching
- L'Hospitals rule and indeterminate forms
- Roll's and mean value theorems
- Classical inequalities
- tangent, normal lines, curvature and radius of curvature
- Evaluate and involute
- Envelope of a family of curves and osculating curves
- related rates
- optimization problems in geometry, physics and economics

9 . Application of Derivatives in Psychology:

The application of differential calculus to mental phenomena:

Dr. Montague's it was pointed out that we can get a very simple expression does for the 'specious present,' which was found to be $\frac{do}{ds}$, if we denote by **o** the objective and by **s** the subjective elements of a psychosis. The second derivative would determine the time flow. Without considering the important philosophical results of the theory we shall make the following observations about the method.

The author considers the ratio of the increments **Δo** and **Δs**, so which occur in the time **Δt**, and the fraction $\frac{\Delta o}{\Delta s}$ is supposed to approach or attain the limit $\frac{do}{ds}$ It will be of some interest to see what suppositions this statement involves. First of all, it is clear that we have to consider the limit$\frac{\frac{\Delta o}{\Delta t}}{\frac{\Delta s}{\Delta t}}$, because **o** is not an explicit function of **s.**

Though we know little or nothing about the sufficient conditions of differentiability, we can in this case readily indicate the following necessary conditions: (1) **o** and s must be continuous; (2) both must have a differential quotient with regard to t; (3) both differential quotients must be continuous; (4)$\frac{ds}{dt}$must not be zero in the whole-time interval under consideration. It is hard to make those assumptions, nothing about the character of the functions dealt since s is apparently discontinuous in many points submitted to the well-known tests.

It is evident that the author had in mind to measure a time period by its relation to a standard change and so to get rid duration, but he did not see that the conditions of the problem became so much more complicated by the implicit relation of o and All these tacit presuppositions would have become clear if the author had assumed that o is an explicit function of s, but such but such a relation, of which we can get no idea, would never have been granted. The establishing of the indirect relation between **o** and **s** by introducing them as functions of time hides the difficulty but does not remove it.

An example will show to what kind of conclusions we come, if we accept the author's view.$\frac{do}{ds}$ varies with time and we may pick out two moments for which this ratio has the same value, as it is always possible because $\frac{do}{ds}$ is continuous and $\frac{d^2o}{ds^2}$ changes sign. The conditions of Rolle's theorem are fulfilled, since continuity of $\frac{do}{ds}$ and existence of the second derivative are supposed by the author, and therefore, the second derivative vanishes at least once. The vanishing of $\frac{d^2o}{ds^2}$ is characteristic for the state of ennui and the first conditions are approximately fulfilled if one sits in a quiet room and recalls something. It follows that one must be bored before one can recall anything. Psychological laws of this kind can be deduced easily by every mathematician.

There is not the least doubt that the whole theory of functions could be applied to a psychology of this kind, but the question remains, whether the conclusions logically deduced from our system admit of a verification by experiment. If we consider it an important feature of experimental psychology, that to every implication of our system corresponds an empirical fact and if possible, vice versa, we must renounce speculations about functions of which we know nothing.

Now supposing for a moment that there are no gaps and errors in the author's proof, could we deduce anything from his laws? Of course not. The function is totally unknown and we must measure empirically the value of $\frac{d^2o}{ds^2}$ It would be important to know the derivative if we could construct the function or if we could verify it in some other way, but as we cannot we must con elude that the use of symbols of which the and the meaning too general is of little help Finally it may be mentioned that the interesting attempt to measure a time period by the ratio of a change occurring in it to a standard change also occurring in it fails, because this ratio is a number which becomes a time only when multiplied by a time unit. For such a standard we choose a certain amount of change in o, to which we refer as a standard, for instance the movement of a pendulum. One of the principal features of a standard is constancy, and measurement is impossible without it. We have therefore either a measurement which varies with time or our whole speculations about the specious present break down, because the differential quotient of a constant vanishes everywhere.

Some Other Applications of Derivatives:

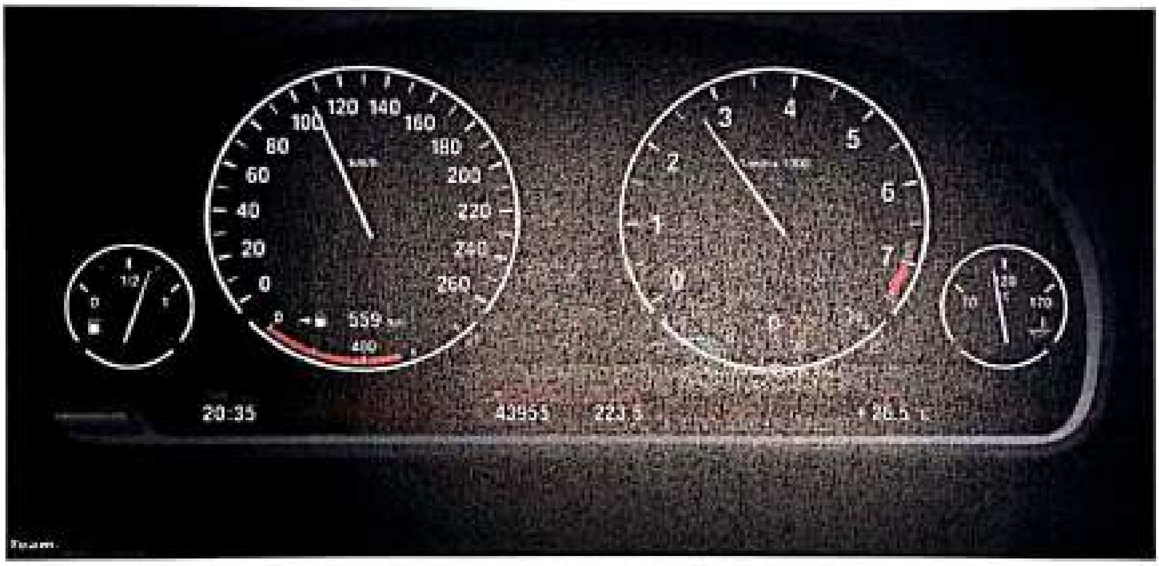
Derivatives are also use to calculate:

- Rate of heat flow in Geology.
- Rate of improvement of performance in psychology
- Rate of the spread of a rumor in sociology.

10. Real Life Applications of Derivatives:

10.1 Automobiles:

In an automobile there is always an odometer and a speedometer. These two gauges work in tandem and allow the driver to determine his speed and his distance that he has traveled. Electronic versions of these gauges simply use derivatives to transform the data sent to the electronic motherboard from the tires to miles per Hour (MPH) and distance (KM).



10.2 Radar Guns:

Keeping with the automobile theme from the previous slide, all police officers who use radar guns are actually taking advantage of the easy use of derivatives. When a radar gun is pointed and fired at your care on the highway. The gun is able to determine the time and distance at which the radar was able to hit a certain section of your vehicle. With the use of derivative, it is able to calculate the speed at which the car was going and also report the distance that the car was from the radar gun.

10.3 Business:

In the business world there are many applications for derivatives. One of the most important application is when the data has been charted on graph or data table such as excel. Once it has been input, the data can be graphed and with the applications of derivatives you can estimate the profit and loss point for certain ventures.

REFRENCES: -

1. Calculus volume-1, second edition by, tom m. Apostol
2. http://www.webmd.com/a-to-2-guides/benign-tumors-causes-treatments#1
3. https://www.blood presureuk.org/ Bloodpresureand you/your body/Arteries
4. https://www.math.hawaii.edu/math499/extracredit
5. https://web.ma.utexas.edu/currentweb
6. https://www.zweigmedia.com/calcsumms
7. https://www.jstor.org/stable/2011945